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SATURATION AND INVERSE THEOREMS FOR COMBINATIONS OF BERNSTEIN-TYPE OPERATORS

by

© CHUNG-PING MAY

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ....

"SATURATION AND INVERSE THEOREMS FOR COMBINATIONS

OF BERNSTEIN-TYPE OPERATORS"

submitted by Chung-Ping MAY

in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

In this thesis, we study the local saturation and inverse problems for Bernstein-type operators $S_\lambda(f,t)$, which include the Bernstein polynomials, Szasz operators, Post-Widder operators, Phillips operators and Baskakov operators, under the combinations defined by

$$(1)\qquad \begin{cases} S_\lambda(f,k,t) = (2^k-1)^{-1}[2^k S_{2\lambda}(f,k-1,t)-S_\lambda(f,k-1,t)] \\ S_\lambda(f,0,t) = S_\lambda(f,t) \end{cases}.$$

The saturation problems are investigated in Chapters II and III, while the inverse problems are investigated in Chapter IV. A local saturation problem is that of determining the class of functions $f(t)$ for which the optimal rate of convergence $S_\lambda(f,t) \to f(t)$ is achieved for f in some interval. A local inverse problem is that of determining the smoothness of functions f for which the rate of convergence of $S_\lambda(f,t) \to f(t)$ is slower than the optimal rate.

In Chapter II, local saturation results for Bernstein polynomials, Szasz operators, Post-Widder operators and Phillips operators, under combinations given in (1), have been achieved. We prove that, if $||S_\lambda(f,k,t)-f(t)||_{C[a,b]} = 0(n^{-(k+1)})$, then $f^{(2k+2)} \in L_\infty[a,b]$.

In Chapter III, generalizations in two directions are made: firstly, we prove a local saturation result for Baskakov operators under combinations (1); secondly, we define a new type of combination

which is more general than the combination (1). The saturation result for Baskakov operators is similar to the results achieved in Chapter II. The new combinations that we defined contain the combination (1) as a special case. However, a different special case for combinations of Bernstein polynomials can be constructed, for which a polynomial with the same degree that in (1) would yield optimal rate of $n^{-(k+1)}$ will have the optimal rate n^{-2^k}.

The proof of the saturation result for the new combinations needs results from Chapter IV, where we study the inverse problem for Bernstein-type operators in the "general" combinations (the new combinations). Using this result as an intermediate step, we achieve the saturation result for the general combinations.

Chapter V is devoted to the saturation and inverse problems for combinations of some exponential formulae of semigroups of operators. Using similar techniques as in the previous chapters, saturation and inverse results for the exponential formulae of Szasz, Kendall, Post-Widder and Phillips are obtained.

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CHAPTER I

INTRODUCTION

The theory of approximation is an old and important subject in analysis. Known results are very rich and exhaustive, yet many problems are still unsolved.

In the theory of approximation, one of the basic problems is to approximate a given function f in X either by functions from some preassigned classes $\mathcal{S}_\lambda$ $(\mathcal{S}_\lambda \subset X)$, or by functions $S_\lambda(f,\cdot)$ "constructed" from f. In the first situation, one considers nested classes $\{\mathcal{S}_\lambda;\ \lambda \in \Lambda\}$ such that $\bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda$ is dense in X, and f is approximated by elements of $\mathcal{S}_\lambda$. For example, $\mathcal{S}_n$ are subspaces of all trigonometric or algebraic polynomials of degree n. In the second situation, the approximation processes $\{S_\lambda(f,\cdot)\}$ are families of linear operators on f such that $S_\lambda(f,x)$ converges to $f(x)$ in some topology. Fejer operators $\sigma_n(f,\cdot)$ and Bernstein polynomials $B_n(f,\cdot)$ defined below are approximation processes of this type.

$$\sigma_n(f,t) = \frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} f(t-u) \left[\frac{\sin \frac{(n+1)u}{2}}{\sin \frac{u}{2}} \right]^2 du, \quad f\ 2\pi \text{ periodic}; \tag{1.1}$$

$$B_n(f,t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right) . \tag{1.2}$$

A natural problem is estimating the rate of convergence of $\|f - S_\lambda(f,\cdot)\| \to 0$, or $\operatorname{dist}(f,\mathcal{S}_\lambda) \to 0$, as $\lambda \to \infty$. Probably the first discussions of problems of this type were by Lebesgue [37] and

de La Vallee Poussin [59] in 1908. A pioneering theorem of D. Jackson [34] in 1911 in this direction shows the connection of the smoothness (modulus of continuity) of the given function f and the "error" (rate of convergence) of dist (f,P_n), where P_n is the space of trigonometric polynomials of degree n.

A Jackson-type theorem is a result that determines the rate of convergence of $||f-S_\lambda(f,\cdot)||$ (resp. dist (f,S_λ)) in terms of properties of the given function f. A Jackson-type theorem is also called a direct theorem. For most approximation processes, the rate of convergence is faster for smoother functions.

A direct theorem for Bernstein polynomials $B_n(f,x)$ was proved in 1935 by T. Popoviciu [47], who showed that, for $f \in C[0,1]$, $|f(x) - B_n(f,x)| \leq \frac{5}{4}\,\omega(n^{-\frac{1}{2}})$, where $\omega(h) = \sup\{|f(x)-f(x+t)|;\ |t| \leq h,\ x,\ x+t \in [0,1]\}$ is the modulus of continuity for f.

In 1932, E. Voronovskaja [60] showed that, if $f \in C^2[0,1]$, then $\lim_{n\to\infty} n[B_n(f,x)-f(x)] = \frac{x(1-x)}{2} f''(x)$. From this asymptotic result, we see that even for $f \in C^\infty[0,1]$, the rate of convergence of $B_n(f,x)$ would not be faster than that of functions in C^2. This leads to the concept of saturation introduced by J. Favard [24].

The saturation problem is that of determining the family $\psi(\lambda)$, $\psi(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$, and the class, called the saturation class or Favard class, of functions f such that $||f-S_\lambda(f,\cdot)|| = O(\psi(\lambda))$ (or dist $(f,S_\lambda) = O(\psi(\lambda))$), but, except for "trivial" classes of functions, the order of convergence, $\psi(\lambda)$, cannot be improved. The saturation result is also referred to as the "optimal case" because the optimal rate of convergence is treated.

The inverse problem (as being the inverse direction of the direct problem), called also the non-optimal case, is that of determining the class of functions f such that $\|f-S_\lambda(f,\cdot)\| = O(\psi'(\lambda))$ (resp. dist $(f,S_\lambda) = O(\psi'(\lambda))$), where $\psi'(\lambda)$ tends to zero slower than the optimal rate $\psi(\lambda)$, and is usually chosen to be $\psi^\alpha(\lambda)$, $0 < \alpha < 1$.

In the literature, most saturation and inverse results for operators are on positive operators. Because of the Korovkin theorem, the convergence of a sequence of linear operators is determined by the functions 1, x and x^2, which relate to the first three terms of Taylors expansion and hence one cannot have a faster convergence than that of C^2 functions. Therefore, the saturation classes for positive operators would generally contain smoothness properties up to having second derivatives. In order to obtain more efficient approximation operators, one has to consider non-positive linear operators for which faster rates of convergence may be obtained with saturation classes related to functions having higher derivatives.

In this thesis, we investigate some non-positive approximation processes obtained by certain linear combinations of some classical operators, e.g., Bernstein polynomials $B_n(f,t)$, Szasz operators $S_\lambda^1(f,t)$, Post-Widder operators $S_n^2(f,t)$ and Phillips operators $S_\lambda^3(f,t)$, defined by:

(1.3) $$S_\lambda^1(f,t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f(\tfrac{k}{\lambda}) \quad ,$$

(1.4) $$S_n^2(f,t) = \frac{1}{(n-1)!} (\tfrac{n}{t})^n \int_0^\infty e^{-nu/t} u^{n-1} f(u)du \quad ,$$

and

$$(1.5) \qquad S_\lambda^3(f,t) = \int_0^\infty e^{-\lambda(t+u)} \sum_{n=1}^\infty \frac{(\lambda^2 t)^n u^{n-1}}{n!(n-1)!} f(u)du + e^{-\lambda t} f(0) \quad .$$

Baskakov operators are also treated in Chapter III. Most of the results we achieved stem from similar results for Bernstein polynomials.

A saturation result for Bernstein polynomials was first achieved by K. de Leeuw [12] in 1959. This result was later refined by G.G. Lorentz [41], and by B. Bajšanski and R. Bojanic [2]. Both the technique used by Lorentz and that of Bajšanski-Bojanic have been extensively developed and generalized. For the applications and extensions of Bajšanski-Bojanic's parabola technique, we refer the reader to the works of V.G. Amelkovic [1], G. Muhlbach [46], G.G. Lorentz and L. Schumaker [42], H. Berens [5] and R. DeVore [15]. Lorentz's technique has also been developed by L. J. DeLuca, K. Ikeno, Y. Suzuki and S. Watanabe [13], [33], [53]-[55]. In this thesis we shall use the technique of Lorentz rather than the parabola technique.

Lorentz's saturation theorem for Bernstein polynomials is a global result. He proved that, for $f \in C[0,1]$, $f' \in \mathrm{Lip}\ 1$ if and only if $|B_n(f,x)-f(x)| \le M \frac{x(1-x)}{n}$. In this thesis we shall achieve <u>local</u> saturation and inverse theorems for certain linear combinations $S_\lambda(f,k,t)$ of some "Bernstein-type" operators $S_\lambda(f,t)$ (which contain the operators (1.2), (1.3), (1.4) and (1.5) as well as Baskakov operators). The local problem is to determine the smoothness of f in some interval from the rate of convergence of

$\sup |S_\lambda(f,k,t)-f(t)|$, where the supremum is taken on the same interval.

Chapter II, Section 1 is devoted to the study of the local saturation problem for $B_n(f,k,t)$, which is a linear combination of Bernstein polynomials $B_n(f,t)$ defined by P.L. Butzer [8] as

$$(1.6)\qquad \begin{cases} B_n(f,k,t) = (2^k-1)^{-1}[2^k B_{2n}(f,k-1,t)-B_n(f,k-1,t)] \\ B_n(f,0,t) = B_n(f,t) \end{cases} .$$

The rate of convergence of $B_n(f,k,t)$ has been investigated by Butzer in [8]. The saturation result we obtained is for $f \in C[0,1]$, $\|B_n(f,k,t)-f(t)\|_{C[a,b]} = 0(\frac{1}{n^{k+1}})$ implies $f^{(2k+2)} \in L_\infty[a,b]$, where $0 < a < b < 1$ (c.f., Theorem II.1.2; see also [21]).

In Section 2 of Chapter II, we investigate the saturation classes for a number of classical approximation processes $S_\lambda(f,t)$, (e.g., Szasz, Post-Widder and Phillips operators defined in (1.2)), under the combinations $S_\lambda(f,k,t)$ similar to $B_n(f,k,t)$ (c.f., equation (2.20) in Chapter II). We found that, in spite of their very different appearances, these operators and the Bernstein polynomials are, from the point of view of rates of convergence, very similar operators. In fact, we found that the rates of convergence of $S_n((x-t)^i,t) \to 0$, $i = 1,2,3,\ldots$, as $n \to \infty$ for each of these operators are the same as $B_n((x-t)^i,t) \to 0$ for Bernstein polynomials (c.f., Lemmas II.1.8 and II.2.8). This property essentially determines the correspondence of the smoothness of the function with the rate of convergence of $S_\lambda(f,t) \to f(t)$. We therefore refer to these operators and the Bernstein polynomials as the Bernstein-type operators. For

some of these operators, for example (1.4), (1.5), no saturation theory even for the positive case $k = 0$ was known.

In Chapter III generalizations in two different directions are discussed. In the first section we investigate the saturation problem for the combinations of Baskakov operators (the definition of Baskakov operators will be given in Chapter III, Definition III.1.1). The construction of the Baskakov operators is a general setting for Bernstein-type operators. The Baskakov operators, which include the Bernstein and Szasz operators, are also Bernstein-type operators. However, the results we have achieved for Baskskov operators (c.f., Theorem III.1.2) are only satisfied for functions with growth not faster than some polynomials while the results for Szasz operators (c.f., Theorem II.2.1) are satisfied for functions with growth not faster than some exponential functions. Nevertheless, our restriction is still less than the one used by Baskakov ([3], p. 250) and Suzuki ([54], pp. 430, 431 and 441). Baskakov's convergence theorem and Suzuki's saturation theorem were only for bounded functions and for functions with compact supports respectively.

In the second section of Chapter III, we have solved a problem raised by Butzer: in the paper ([8], p. 567) he asked whether there exist any linear combinations for Bernstein polynomials other than the one discussed there (i.e., (1.3)), such that it is a polynomial of the same degree as $B_n(f,k,t)$ but with faster optimal rates of convergence. We solved this problem by defining a more general combination, such that the combination investigated by Butzer is a special case. Butzer defines the combinations by induction

(i.e., by equation (1.3)); while we define the combinations by the explicit formulae for the coefficients:

$$(1.7)\qquad \begin{cases} S_\lambda(f,k,t) = \sum_{j=0}^{k} C(j,k) S_{d_j\lambda}(f,t) \ , \\ C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \dfrac{d_j}{d_j - d_i} \ . \end{cases}$$

The coefficients depend on $k+1$ arbitrarily chosen distinct positive integers $d_0,\ldots,d_k$. With the proper choice of these integers (viz. $d_i = i+1$), formula (1.7) would define a polynomial of degree $(k+1)n$ with the optimal rate of convergence $n^{-(k+1)}$ (c.f., Remark III.2,5) while for that rate Butzer needed a polynomial of degree $2^k n$.

The saturation problem for Bernstein-type operators under the "general" combinations (1.4) has also been investigated and a similar saturation result is also achieved. However, the proof of this theorem has to be delayed until the end of Chapter IV after we have achieved the inverse results for the "general" combinations of the Bernstein-type operators.

Chapter IV is devoted to the inverse problem for Bernstein-type operators (for the "general" combinations). In the literature the starting point of the inverse theorems follows an inequality proved by Bernstein [7] in 1912 for periodic functions approximated by trigonometric functions, and analogously estimates of $\left\| \frac{d^p}{dx^p} S_\lambda(f,x) \right\|$ play the important roles in the proof of the inverse theorem here. An inverse theorem for Bernstein polynomials was first proved by H. Berens and G.G. Lorentz in 1972 [5] (see Theorem IV.1.2). They

applied the recently developed K-interpolation method of J. Peetre (c.f., e.g., ([10], p. 165)) to solve the problem. According to their remark ([5], p. 693), an elementary method has also been tried but they could not obtain a complete solution by such a method. R. DeVore had also attempted this problem and obtained some weaker result some time ago (c.f., the remark in [5], p. 694). After Berens and Lorentz's result DeVore solved the inverse theorem for Bernstein polynomials by using the parabola technique (see [14] and also [15]).

We solve the local inverse problems of the combinations discussed above following the Berens-Lorentz method. In Theorem IV.1.4 we proved an inverse theorem for a number of Bernstein-type operators (e.g., Bernstein polynomials, the operators of Szasz, Post-Widder and Baskakov) under the "general" combinations. We should, however, particularly mention here, that we are not able to prove the inverse theorem for Phillips operators (1.5).

By using the inverse theorem as an intermediate result, we proved the saturation theorem of "general" combinations of Bernstein-type operators in this chapter (Theorem IV.5.1). This theorem also includes the Phillips operators (1.5).

As an application, in Chapter V we show a saturation and an inverse theorem for linear combinations of exponential formulae for semigroups of operators (Theorems V.3.1 and V.3.2).

We would like to emphasize here that an attempt to use elementary methods has been made through all the proofs. Therefore, in some places, more than one proof has been given.

CHAPTER II

SATURATION THEOREMS FOR COMBINATIONS OF BERNSTEIN-TYPE OPERATORS

Saturation theory is an important part of approximation theory. It is the study of the problem of determining the class of functions for which a given approximation process has the optimal order. The concept of saturation was first introduced by J. Favard in 1949 for summation methods of Fourier series [24].

For Bernstein polynomials

$$B_n(f,t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f(\tfrac{k}{n})$$

the saturation result was proved by de Leeuw [12], Bajsanski and Bajanic [2] and Lorentz [41]. The method of proof used by Lorentz and that of Bajsanski and Bojanic have also been applied to other approximation processes [33], [53]-[55].

After obtaining the saturation class for a known approximation process, the following questions arise naturally: Is it possible to construct from well-known operators new ones which approximate the original functions with possibly faster rate? If so, how does one construct such operators, and, can one determine their saturation classes as well?

Butzer [8] approached these problems by using the following linear combinations as the new approximation process:

$$(2.1)\qquad \begin{cases} B_n(f,k,t) = (2^k-1)^{-1}[2^k B_{2n}(f,k-1,t)-B_n(f,k-1,t)] \\ B_n(f,0,t) = B_n(f,t) \quad . \end{cases}$$

The rates of convergence for such combinations have also been investigated, not only for Bernstein polynomials (see e.g., [8] and [20]), but also for other operators (e.g., for Gamma-Operators, see [38], where they investigated the case when $k = 2$). In this chapter, we shall investigate the local saturation problem for such linear combinations formed from a number of classical operators. One should note however that Butzer's direct theorem result will also be extended here.

§1. The Saturation Result for Bernstein Polynomials

We shall prove the saturation theorems for Bernstein polynomials in this section.

Definition II.1.1

An approximation process S_λ on a space X is called "saturated" if there exists an "optimal" order $\psi(\lambda)$, where $\psi(\lambda) > 0$ and $\lim_{\lambda\to\infty} \psi(\lambda) = 0$, and a degenerate class of functions $L(S_\lambda)$, such that

(i) $||S_\lambda(f,\cdot)-f(\cdot)|| = o(\psi(\lambda))$ if and only if $f \in L(S_\lambda)$;

(ii) there is a function $f_o \in L(S_\lambda)$ such that

$||S_\lambda(f_o,\cdot)-f_o(\cdot)|| = O(\psi(\lambda))$.

The saturation result for Bernstein polynomials will be given in the following theorem.

Theorem II.1.2

Let $f \in C[0,1]$, $0 < a < a_1 < b_1 < b < 1$, $n_i = n_o 2^i$, and $I(f,n,k,a,b) = n^{k+1} \|B_n(f,k,\cdot)-f(\cdot)\|_{C[a,b]}$. Then in the following (1) ⟹ (2) ⟹ (3) and (4) ⟹ (5) ⟹ (6):

(1) $I(f,n_i,k,a,b) = O(1)$, $n_i \to \infty$;

(2) $f^{(2k+1)} \in A.C.(a,b)$ and $f^{(2k+2)} \in L_\infty[a,b]$

(3) $I(f,n,k,a_1,b_1) = O(1)$, $n \to \infty$;

(4) $I(f,n_i,k,a,b) = o(1)$, $n_i \to \infty$;

(5) $f \in C^{2k+2}(a,b)$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t)f^{(i)}(t) = 0$, $t \in (a,b)$, where $Q(i,k,t)$ are polynomials depending on k;

(6) $I(f,n,k,a_1,b_1) = o(1)$, $n \to \infty$.

We shall prove this theorem by induction. In case $k = 0$, this theorem has been proved by de Leeuw in [12]. However, we would like to point out that the proof given below also can be used to prove the case $k = 0$ directly.

§1.1 Outline of the Proof of the Bernstein Polynomials Saturation Theorem.

Since the proof is long and intertwining, we shall outline the proof in this subsection and give the details in the following subsections. There are seven basic steps in our proof.

(I) We first observe that, for each k and any bounded function f continuous at t, the relation $\lim_{n\to\infty} B_n(f,k,t) = f(t)$ holds by applying the defining relation (1.1). We use the following lemma to reduce the saturation result of $n^{k+1}||B_n(f,k,\cdot)-f(\cdot)|| = O(1)$ (or $o(1)$) to that of $n^{k+1}||B_{2n}(f,k,\cdot)-B_n(f,k,\cdot)|| = O(1)$ (or $o(1)$).

Lemma II.1.3

If $f \in C[0,1]$ and $n_i = 2^i n_0$, then in the following (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3):

(1) $n_i^{k+1}||B_{n_i}(f,k,\cdot)-f(\cdot)||_{C[a,b]} \le M$, for all i;

(2) $n_i^{k+1}||B_{2n_i}(f,k,\cdot)-B_{n_i}(f,k,\cdot)||_{C[a,b]} \le 2M$, for all i;

(3) $n_i^{k+1}||B_{n_i}(f,k,\cdot)-f(\cdot)||_{C[a,b]} \le 4M$, for all i.

(II) Using Lemma II.1.3, we have $n_i^{k+1}||B_{2n_i}(f,k,\cdot)-B_{n_i}(f,k,\cdot)||_{C[a,b]} \le M$. Since for any $g \in C_o^\infty$ such that $\operatorname{supp} g \subset (a,b)$, we have $g \in L_1[a,b]$, the dual space of which is $L_\infty[a,b]$, then by means of Alaoglu's theorem, there exists an $h \in L_\infty[a,b]$ and a subseqeunce $\{n_{i_\rho}\}$ of $\{n_i\}$ such that for any g as above, we have

$$(2.2) \qquad \langle n_i^{k+1}[B_{2n_{i_\rho}}(f,k,\cdot)-B_{n_{i_\rho}}(f,k,\cdot)],\ g(\cdot)\rangle \to \langle h(\cdot),g(\cdot)\rangle\ .$$

(III) In an effort to investigate the expression (2.2), we derive an asymptotic relation for $n^{k+1}[B_{2n}(f,k,\cdot)-B_n(f,k,\cdot)]$ provided that $f \in C^{2k+2}$.

Lemma II.1.4

Let $f \in C[0,1]$. If $f^{(2k+2)}(t)$ exists, then

$$(2.3) \qquad n^{k+1}[B_{2n}(f,k,t)-B_n(f,k,t)] = \sum_{j=k+1}^{2k+1} Q(j,k,t)f^{(j)}(t)+o(1)$$

$$\equiv P_{2k+2}(D)f(t)+o(1)$$

where $Q(j,k,t)$ are polynomials in t and also

$$Q(2k+2,k,t) = c_1[t(1-t)]^{k+1}, \qquad Q(2k+1,k,t) = c_2[t(1-t)]^k(1-2t) .$$

Moreover, if $f \in C^{2k+2}[a,b]$, then (2.3) is uniform on every interior subinterval $[a_1,b_1] \subset (a,b)$.

(IV) As a consequence of Lemma II.1.4, we have for $f \in C^{2k+2}[a,b]$, $g \in C_o$ with supp $g \subset (a,b)$, the following relation:

$$(2.4) \qquad \lim_{n_i \to \infty} \langle n_i^{k+1}[B_{2n_i}(f,k,\cdot)-B_{n_i}(f,k,\cdot)], g(\cdot)\rangle$$

$$= \langle P_{2k+2}(D)f(\cdot),g(\cdot)\rangle = \langle f(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle$$

where $P^*_{2k+2}(D)$ is the dual operator of $P_{2k+2}(D)$. (In this case, in fact, it is nothing but a result of integration by parts.)

(V) Now suppose f satifies statement (1) of the theorem. For such f, we shall prove $f^{(2k)} \in L_\infty[a,b]$ as an intermediate result, which will be used in further estimations. This fact follows by the following lemma, and the induction hypothesis that the theorem holds for $k-1$.

Lemma II.1.5

Let $f \in C[0,1]$, $\delta > 0$ and $n_i = n_o 2^i$. Then $n_i^{k+1} \| B_{2n_i}(f,k,\cdot) - B_{n_i}(f,k,\cdot) \|_{C[a,b]} \leq M$ implies

$$n_i^k \| B_{2n_i}(f,k-1,\cdot) - B_{n_i}(f,k-1,\cdot) \|_{C[a,b]} \leq M_1 \quad .$$

Let $L_1^{2k}[a,b] \equiv \{f \in C[0,1], f^{2k} \in L_1[a,b]\}$, equipped with the norm $\|\cdot\|_{L_1^{2k}[a,b]}$ defined by:

$$\|f\|_{L_1^{2k}[a,b]} = \|f\|_{C[0,1]} + \max_{0 \leq i \leq 2k} \|f^{(i)}\|_{L_1[a,b]} \quad .$$

Since $C[0,1] \cap C^{2k+2}[a,b]$ dense in $L_1^{2k}[a,b]$ w.r.t. $\|\cdot\|_{L_1^{2k}}$, there exists a sequence $\{f_\ell\}$ in $C[0,1] \cap C^{2k+2}[a,b]$ converging to f in $\|\cdot\|_{L_1^{2k}}$-norm.

In considering the expression of

$$\text{(2.5)} \qquad \lim_{\ell \to \infty} \lim_{n_i \to \infty} \langle n_i^{k+1}[B_{2n_i}(f_\ell,k,\cdot) - B_{n_i}(f_\ell,k,\cdot), g(\cdot)\rangle$$

for such a sequence f_ℓ, we need the following crucial lemma:

Lemma II.1.6

Let $f \in L_\infty^{2k}[a,b]$, $g \in C_o^\infty$ with supp $g \subset (a,b)$. Then

$$\text{(2.6)} \qquad |n_i^{k+1} \langle [B_{2n_i}(f,k,\cdot) - B_{n_i}(f,k,\cdot)], g(\cdot)\rangle| \leq M \|f\|_{L_1^{2k}[a,b]} \quad .$$

where M depends on g (and its derivatives).

Thus, for $\{f_\ell\} \subset C^{2k+2}[a,b] \cap C[0,1]$, converging to f in $||\cdot||_{L_1^{2k}}$-norm, the limits in (2.5) can be interchanged.

Consequently, if f satisfies (1) in the theorem, it also satisfies (2.4). Combining this and (2.2), we get $\langle h(\cdot),g(\cdot)\rangle =$ $= \langle f(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle$ for all $g \in C_o^\infty$ with supp $g \subset (a,b)$.

This implies $P_{2k+2}(D)f(t) = h(t)$ since they are euqal as generalized functions. However, as a first order linear differential equation for $f^{(2k+1)}$, with the non-homogeneous term, which can be represented in terms of $f^{(i)}$, $i \leq 2k$, and h, in $L_\infty[a,b]$, we deduce that $f^{(2k+1)} \in A.C.[a,b]$, and hence $f^{(2k+2)} \in L_\infty[a,b]$.

(VI) The "little o" part is similar with only one difference: instead of $\langle h(\cdot),g(\cdot)\rangle = \langle f(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle$, we have $\langle f(\cdot),P^*_{2k+2}(D)g(\cdot)\rangle = 0$.

(VII) The implications (2) $\Longrightarrow$ (3), and (5) $\Longrightarrow$ (6) in the theorem are slightly stronger than Lemma II.1.4. But as $f^{(2k+1)} \in A.C.[a,b]$, and $f^{(2k+2)} \in L_\infty[a,b]$, we have $f^{(2k+1)} \in \mathrm{Lip}\,(1;a,b)$. The rest of these proofs are computational and will be omitted.

The proofs of the four lemmas stated in this section is given in the following sections.

§1.2 Proof of Some Auxiliary Lemmas (Lemmas II.1.3 and II.1.5)

Proof of Lemma II.1.3

The first implication follows directly from the triangular inequality:

$$n_i^{k+1}\|B_{2n_i}(f,k,\cdot)-B_{n_i}(f,k,\cdot)\|_{C[a,b]}$$

$$\leq 2^{-(k+1)}(2n_i)^{k+1}\|B_{2n_i}(f,k,\cdot)-f(\cdot)\|_{C[a,b]} +$$

$$+ n_i^{k+1}\|B_{n_i}(f,k,\cdot)-f(\cdot)\|_{C[a,b]}$$

$$\leq 2M .$$

For the second implication,

$$n_i^{k+1}\|f(\cdot)-B_{n_i}(f,k,\cdot)\|_{C[a,b]}$$

$$= \lim_{m\to\infty} n_i^{k+1}\|B_{2^m n_i}(f,k,\cdot)-B_{n_i}(f,k,\cdot)\|_{C[a,b]}$$

$$\leq \lim_{m\to\infty} \sum_{\ell=1}^{m} 2^{-(\ell-1)(k+1)}\{(2^{\ell-1}n_i)^{k+1} \cdot$$

$$\cdot\|B_{2^\ell n_i}(f,k,.)-B_{2^{\ell-1}n_i}(f,k,\cdot)\|_{C[a,b]}\}$$

$$\leq 4M .$$

<u>Proof of Lemma II.1.5</u>

Using the recursion relation (2.1) and $n_i = n_o 2^i$, we have

$$I(N) \equiv (2^k-1) \sum_{i=0}^{N} 2^{ki}[B_{2n_i}(f,k,t)-B_{n_i}(t,k,t)]$$

$$= \sum_{i=0}^{N} 2^{ki}\{2^k B_{4n_i}(f,k-1,t)-B_{2n_i}(f,k-1,t)\} -$$

$$- \sum_{i=0}^{N} 2^{ki}\{2^k B_{2n_i}(f,k-1,t)-B_{n_i}(f,k-1,t)\}$$

$$= 2^{k(N+1)}\{B_{2n_{N+1}}(f,k-1,t)-B_{n_{N+1}}(f,k-1,t)\} -$$

$$- 2^k\{B_{2n_o}(f,k-1,t)-B_{n_o}(f,k-1,t)\} .$$

Using the assumption of our lemma,

$$\|I(N)\|_{C[a,b]} \leq (2^k-1) \sum_{i=0}^{N} 2^{ki}(2^i n_o)^{-(k+\delta)} M$$

$$\leq [(2^k-1) \cdot (1-2^{-\delta})^{-1} M]\, n_o^{-(k+\delta)} \quad ,$$

and therefore,

$$\| 2^{k(N+1)} \{B_{2n_{N+1}}(f,k-1,\cdot)-B_{n_{N+1}}(f,k-1,\cdot)\} \|_{C[a,b]}$$

$$\leq [(2^k-1) \cdot (1-2^{-\delta})^{-1}\, M n_o^{-(k+\delta)} + K$$

which concludes the proof.

§1.3 A Recursion Relation

The proof of the other two lemmas requires further results which we prove in this section. These results will also be used in subsequent sections.

First we define the following notation:

(2.7) $$W(n,t,u) = \sum_{m=0}^{n} \binom{n}{m} (1-t)^{n-m} \delta(u - \frac{m}{n})$$

$$B_n(f,t) = \int_0^1 W(n,t,u) f(u) du$$

where $\delta(\cdot)$ is the Kronecker δ-function.

Lemma II.1.7

For $W(n,t,u)$ defined by (1.7),

(2.8) $$\frac{\partial}{\partial t} W(n,t,u) = \frac{n}{p(t)} W(n,t,u)(u-t)$$

where $p(t) = t(1-t)$.

Proof

$$\frac{\partial}{\partial t} W(n,t,u) = \sum_{m=0}^{n} \binom{n}{m} t^m (1-t)^{n-m} \delta(u - \frac{m}{n}) \frac{m}{t} -$$

$$- \sum_{m=0}^{n} \binom{n}{m} t^m (1-t)^{n-m} \delta(u - \frac{m}{n}) \frac{n-m}{1-t}$$

$$= \frac{n}{t} W(n,t,u)\, u - \frac{n}{1-t} W(n,t,u) + \frac{n}{1-t} W(n,t,u) u$$

$$= \frac{n}{t(1-t)} W(n,t,u)(u-t) \quad .$$

Lemma II.1.8

Let $A_m(n,t)$ be given by

(2.9) $$A_m(n,t) = n^m \int_0^1 W(n,t,u)(u-t)^m du , \qquad m = 0,1,2,\ldots ,$$

then

(a) $A_{m+1}(n,t) = mnp(t)A_{m-1}(n,t)+p(t) \frac{d}{dt} A_m(n,t)$;

(b) $A_m(n,t)$ is a polynomial in t and n ;

(c) The degree of $A_m(n,t)$ in n is $[\frac{m}{2}]$;

(d) The coefficient of n^m in the polynomial $A_{2m}(n,t)$ is $c_1\, p(t)^m$ and in the polynomial $A_{2m+1}(n,t)$ is $c_2\, p^m(t)p'(t)$.

Proof

The recursion formula (a) follows from Lemma II.1.7. The

rest are derived from (a) and induction.

§1.4 The Voronovskaja-Type Relation

Using the formulae established in the last section, we prove Lemma II.1.4 in this section.

To begin with, by using the recursion relation (2.1), we obtain

$$(2.10)\qquad \begin{cases} B_n(f,k,t) = \sum_{j=0}^{k} C(j,k)B_{2^j n}(f,t) \\ \\ B_{2n}(f,k,t)-B_n(f,k,t) = \sum_{j=0}^{k+1} \alpha(j,k)B_{2^j n}(f,t) \end{cases} .$$

Obviously, $C(j,k)$ and $\alpha(j,k)$ are constants that depend only on (2.1), and satify $\sum_{j=0}^{k} C(j,k) = 1$. Moreover, we have the following property for $\alpha(j,k)$, which will be useful in proving Lemmas II.1.4 and II.1.6.

Lemma II.1.9

Let $\alpha(j,k)$ be defined by (2.10), then

$$(2.11)\qquad \sum_{j=0}^{k+1} \alpha(j,k)2^{-mj} = 0 \ , \quad \text{for} \quad m = 0,1,\ldots,k \ .$$

Proof

It is easily seen that $\alpha(0,0) = -1$, $\alpha(1,0) = 1$, $\alpha(0,1) = 1$, $\alpha(1,1) = -3$ and $\alpha(2,1) = 2$, and therefore (2.11) is valid for $k = 0$ and $k = 1$. We proceed by induction. On one hand,

$$(2^k-1)[B_{2n}(f,k,t)-B_n(f,k,t)]$$

$$= (2^k-1) \sum_{j=0}^{k+1} \alpha(j,k)B_{2^j n}(f,t)$$

while by the relation (2.1) together with (2.10),

$$(2^k-1)[B_{2n}(f,k,t)-B_n(f,k,t)]$$

$$= 2^k \sum_{j=0}^{k} \alpha(j,k-1)B_{2^{j+1}n}(f,t) - \sum_{j=0}^{k} \alpha(j,k-1)B_{2^j n}(f,t) .$$

Therefore, $(2^k-1)\alpha(j,k) = 2^k\alpha(j-1,k-1)-\alpha(j,k-1)$ for $1 \le j \le k$, $(2^k-1)\alpha(0,k) = -\alpha(0,k-1)$, and $(2^k-1)\alpha(k+1,k) = 2^k\alpha(k,k-1)$. Thus,

$$(2^k-1) \sum_{j=0}^{k+1} \alpha(j,k)2^{-jm} = 2^k \sum_{j=1}^{k+1} \alpha(j-1,k-1)2^{-jm} -$$

$$- \sum_{j=0}^{k} \alpha(j,k-1)2^{-jm} = 0$$

for $m = 0,1,\ldots,k-1$, by the induction hypothesis. For $m = k$, we have

$$2^k \sum_{j=1}^{k+1} \alpha(j-1,k-1)2^{-jk} - \sum_{j=0}^{k} \alpha(j,k-1)2^{-jk} = 0 \quad .$$

<u>Proof of Lemma II.1.4</u>

Suppose $f^{(2k+2)}(t)$ exits. By (2.10), (2.7), and Taylor's expansion, we have

$$n^{k+1}[B_{2n}(f,k,t)-B_n(f,k,t)]$$

$$= n^{k+1}\sum_{j=0}^{k+1}\alpha(j,k)\int_0^1 W(2^j n,t,u)\left[\sum_{m=0}^{2k+2}\frac{f^{(m)}(t)}{m!}(u-t)^m + \right.$$

$$\left. + \varepsilon(u,t)(u-t)^{2k+2}\right]du$$

$$= I_1 + I_2 \quad ,$$

where $\varepsilon(u,t) \to 0$ as $u \to t$; also, $\varepsilon(u,t)$ is bounded.

First, we calculate I_1. It follows from Lemma II.1.8 that $n^{-m}A_m(n,t) = \int_0^1 W(n,t,u)(u-t)^m du$ is a polynomial in $\frac{1}{n}$ and t. Moreover, following the same lemma, we see that the highest order of $\frac{1}{n}$ is m, while the lowest order is $[\frac{m}{2}+1]$. Thus, by Lemma II.1.9, we have $\sum_{j=0}^{k+1}\alpha(j,k)\int_0^1 W(2^j n,t,u)(u-t)^m du = 0$ for $m \le k$, and for $m \ge k+1$, $\sum_{j=0}^{k+1}\alpha(j,k)\int_0^1 W(2^j n,t,u)(u-t)^m du$ is equal to $(\frac{1}{n})^{k+1}$ multiplying a polynomial in $(\frac{1}{n})$ and t. In other words, from Lemma II.1.8 we obtain

$$(2.12) \qquad I_1 = \sum_{j=k+1}^{2k+2} Q(j,k,t)f^{(j)}(t) + o(1)$$

$$\equiv P_{2k+2}(D)f(t) + o(1) \quad , \quad \text{as} \quad n \to \infty \; .$$

where $Q(2k+2,k,t) = (2k+2)!![t(1-t)]^{k+1}$, and $Q(2k+1,k,t) = c_2[t(1-t)]^k(1-2t)$.

We have to show $I_2 = o(1)$. For this end, we estimate the typical expression:

$$J_n = n^{k+1}\int_0^1 W(n,t,u)\varepsilon(u,t)(u-t)^{2k+2}du \quad .$$

Let $\varepsilon > 0$ be given. Choose a $\delta = \delta(\varepsilon,t)$, such that when $|u-t| < \delta$, we have $|\varepsilon(u,t)| < \varepsilon$. Then, we get:

$$|J_n| \le n^{k+1} \int_{|u-t|<\delta} W(n,t,u)\,\varepsilon(u,t)(u-t)^{2k+2}\,du$$

$$+ n^{k+1} \int_{|u-t|\le\delta} W(n,t,u)\,\varepsilon(u,t)(u-t)^{2k+2}du$$

$$= I_3 + I_4 \quad ,$$

$$I_3 \le \varepsilon\, n^{k+1} \int_{|u-t|<\delta} W(n,t,u)(u-t)^{2k+2}\,du$$

$$\le M_1\varepsilon \;,$$

$$I_4 \le \frac{1}{\delta^2}\, n^{k+1}||\varepsilon(u,t)|| \int_{|u-t|\ge\delta} W(n,t,u)(u-t)^{2k+4}du$$

$$\le \frac{M_2}{\delta^2} \cdot n^{-1} \quad .$$

Hence, $\overline{\lim\limits_{n \to \infty}} |J_n| \le M_1\varepsilon$. Since ε is arbitrary, we have $\overline{\lim\limits_{n \to \infty}} |J_n| = 0$. In other words, $I_2 = o(1)$ as $n \to \infty$. This proves relation (2.3).

When $f \in C^{2k+2}[a,b]$, $f^{(2k+2)}$ is uniform continuous in $[a,b]$. Let $[a_1,b_1] \subset (a,b)$. We observe for $t \in [a_1,b_1]$:

(i) the little o in (2.12) is uniform (in fact, it equals to a polynomial in $\frac{1}{n}$ and t) ;

(ii) in estimating I_2, we can choose $\delta \le \min\,\{a_1-a, b-b_1\}$ independent of t (by the uniform continuity of $f^{(2k+2)}$ in $[a,b]$),

and so the uniformity of (2.3) follows.

§1.5 The Crucial Step

Lemma II.1.6 which in our opinion is crucial for the proof of this theorem will be proved here.

Let $f \in L_1^{2k}[a,b]$, and $g \in C_o^\infty$ with supp $g \subset (a,b)$. By (2.10) and Taylor's expansion, we can estimate $n^{k+1}\langle[B_{2n}(f,k,\cdot)-B_n(f,k,\cdot)],g(\cdot)\rangle$ as follows:

$$(2.13) \qquad n^{k+1}\langle[B_{2n}(f,k,\cdot)-B_n(f,k,\cdot)],g(\cdot)\rangle$$

$$= n^{k+1}\int_0^1\int_0^1\sum_{j=0}^{k+1}\alpha(j,k)W(2^jn,t,u)f(u)\cdot$$

$$\cdot\sum_{\ell=0}^{2k+2}\frac{1}{\ell!}g^{(\ell)}(u)(t-u)^{\ell}dtdu$$

$$+ n^{k+1}\int_0^1\int_0^1\sum_{j=0}^{k+1}\alpha(j,k)W(2^jn,t,u)f(u)\varepsilon(u,t)(t-u)^{2k+2}dtdu$$

$$\equiv I_1 + I_2 \quad \sum_{\ell=0}^{2k+1} J_\ell + I_2 \ .$$

We estimte I_2 first, using Lemma II.1.8,

$$(2.14) \qquad |I_2| \le \sup_{t,u}|\varepsilon(t,u)f(u)|\cdot\left[\sum_{j=0}^{k+1}|\alpha(j,k)|\right]\cdot$$

$$\cdot\max_j n^{k+1}\int_0^1\int_0^1 W(2^jn,t,u)(t-u)^{2k+2}dtdu$$

$$\le K_1\sup_{t,u}|\varepsilon(t,u)|\,||f||_{C[a,b]} \le K_2 \ .$$

To estimate J_ℓ, we evaluate the following typical expression

$$(2.15)\qquad T_i = n^{k+1} \int_a^b \int_0^1 \sum_{j=0}^{k+1} \alpha(j,k) W(2^j n,t,u)[f(u)g^{(\ell)}(u)u^{\ell-i}]t^i dtdu$$

$$\equiv n^{k+1} \sum_{j=0}^{k+1} \alpha(j,k) \int_0^b \int_0^1 W(n_j,t,u)\phi_i(u)t^i dtdu$$

where $n_j = 2^j n$, and $\phi_i(u) = f(u)g^{(\ell)}(u)u^{\ell-i}$. Note that $\phi_i^{(2k)} \in L_\infty$, with $\operatorname{supp}\phi_i \subset (a,b)$. Since

$$(2.16)\qquad \int_0^1 W(n,t,u)t^i du = \sum_{m=0}^{n} \frac{(m+1)\ \dots\ (m+i)}{(n+1)\ \dots\ (n+1+i)}\ \delta(u - \frac{m}{n}) \quad ,$$

then

$$(2.17)\qquad T_i = n^{k+1} \sum_{j=0}^{k+1} \alpha(j,k) \sum_{\frac{m}{n_j} \in (a,b)} \phi_i(\frac{m}{n_j}) \frac{(m+1)\ \dots\ (m+i)}{(n_j+1)\ \dots\ (n_j+1+i)}$$

$$= n^{k+1} \sum_{j=0}^{k+1} \alpha(j,k) \frac{n_j^{i+1}}{(n_j+1)\ \dots\ (n_j+1+i)} \cdot$$

$$\cdot \sum_{\frac{m}{n_j} \in (a,b)} \frac{1}{n_j} \phi_i(\frac{m}{n_j})(\frac{m}{n_j} + \frac{1}{n_j}) \dots (\frac{m}{n_j} + \frac{i}{n_j}) \quad .$$

Further, we can write

$$\frac{n_j^{k+1}}{(n_j+1)\ \dots\ (n_j+1+i)} = 1 + \frac{d_1}{n_j} + \dots + \frac{d_k}{(n_j)^k} + 0(\frac{1}{n_j^{k+1}})$$

and

$$\left(\frac{m}{n_j}+\frac{1}{n_j}\right)\cdots\left(\frac{m}{n_j}+\frac{i}{n_j}\right)=\left(\frac{m}{n_j}\right)^i+\left[\sum_{\ell=0}^{i-1} e_\ell\left(\frac{m}{n_j}\right)\left(\frac{1}{n_j}\right)^{i-\ell}\right]$$

where neither $d_1,\ldots,d_k$ nor $e_0,\ldots,e_{i-1}$ depends on j.

Using the Euler-Mclaurin formula ([4], pp. 268-275), we obtain

$$(2.18)\qquad \frac{1}{n_j}\sum_{\frac{m}{n_j}\in[a,b]} \phi_i\left(\frac{m}{n_j}\right)\left(\frac{m}{n_j}\right)^i=\int_a^b \phi_i(u)u^i du+R\ ,$$

where, for $y(u)=\phi_i(u)u^i=f(u)(g^{(\ell)}(u)u^\ell)$,

$$R=\frac{1}{n^{2k+1}}\sum_{m=k_o}^{n_o}\int_0^1 P_{2k}(t)y^{(2k)}(a+n_j^{-1}(t+m))dt,\qquad n_o-k_o\le n_j-1\ .$$

Using the fact that $\|P_{2k}\|_{C[0,1]}=(-1)^k P_{2k}(\frac{1}{2})=(-1)^k\left[\frac{B_{2k}(\frac{1}{2})-B_{2k}}{(2k)!}\right]$, where $B_{2k}(\frac{1}{2})$ is the Bernoulli polynomial at $\frac{1}{2}$ and B_{2k} is the Bernoulli number, we estimate R as

$$|R|\le\frac{|P_{2k}(\frac{1}{2})|}{(2k)!\,n_j^{2k+1}}\sum_{m=k_o}^{n_o}\int_0^1 |y^{(2k)}(a+n_j^{-1}(t+m))|dt$$

$$=\frac{|P_{2k}(\frac{1}{2})|}{(2k)!\,n_j^{2k}}\cdot\int_a^b |y^{(2k)}(t)|dt\ .$$

Sine $y^{(2k)}$ can be expressed as polynomial of $f^{(\gamma_1)}g^{(\gamma_2)}u^{\gamma_3}$,

$|y^{(2k)}(t)|\le M\max\{(b-a)^{\gamma_3}\|g^{(\gamma_2)}\|\,|f^{(\gamma_1)}(t)|\ ;\ 0\le\gamma_1\le 2k\ ,$

$\ell\le\gamma_2\le 2k+\ell\ ,\ \max(0,\ell-2k)\le\gamma_3\le\ell,\ \gamma_1+\gamma_2-\gamma_3=2k\}$. In other words, $|R|\le\frac{M}{n_j^{2k}}\|f\|_{L_1^{2k}[a,b]}$.

With this notation, the estimate for T_i is

$$|T_i| \le n_j^{k+1} \left| \sum_{j=0}^{k+1} \alpha(j,k)\left(1 + \frac{d_1}{n_j} + \dots + \frac{d_j}{n_j^k}\right) \cdot \right.$$

$$\left. \cdot \left[\int_a^b \phi_i(u)u^i du + \sum_{\ell=0}^{i-1} \left(\frac{1}{n_j}\right)^{i-\ell} e_\ell \int_a^b \phi_i(u)u^\ell du\right] \right|$$

$$+ n_j^{k+1} \left[\sum_{j=0}^{k+1} |\alpha(j,k)|\right] M' n_j^{-2k} ||f||_{L_1^{2k}[a,b]} + O(1) \quad .$$

The second term is $O(n^{-k+1})$. Recalling Lemma II.1.9, $\sum_{j=0}^{k+1} \alpha(j,k)\frac{1}{n_j^m} = 0$ for $m = 0,1,\dots,k$, and therefore the first term is $O(1)$ at most.

(Q.E.D.)

§1.6 Remarks

1. One cae write Theorem II.1.2 in such a way that the conditions are necessary and sufficient, as is common for saturation results, as the following:

Theorem II.1.2*

Let f be as in Theorem II.1.2. Then in the following (1) $\iff$ (2) and (3) $\iff$ (4):

(1) $I(f,n_i,k,\alpha,\beta) \le M(\alpha,\beta)$, for all $[\alpha,\beta] \subset (a,b)$;

(2) $f^{(2k+1)} \in A.C.(\alpha,\beta)$ and $f^{(2k+2)} \in L_\infty(\alpha,\beta)$, for all $[\alpha,\beta] \subset (a,b)$;

(3) $I(f,n_i,k,\alpha,\beta) = o(1)$, for all $[\alpha,\beta] \subset (a,b)$;

(4) $f \in C^{2k+2}(a,b)$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t)f^{(i)}(t) = 0$ in (a,b).

2. The gap between the necessary and sufficient conditions in Theorem II.1.2, namely, the fact that the conditions are not on the same intervals, though not a big gap, is vital.

When the intervals are changed to be the same, the theorem remains true only for some special cases, for instance, the case of functions with compact supports in the interior of the interval. That is, modification of the end-points, either for functions under consideration or for intervals being dealt with, is necessary. We are not doing any further investigation in this direction.

In the general case, however, the result will not hold if $a_1 = a$ or $b_1 = b$.

The following counter example is "essentially" due to Butzer [4], where he has proved the case $k = 1$ for another purpose.

<u>Example I</u>

Let $0 < \alpha < 2$. There exists a function $f_\alpha \in C^\infty[a,b] \cap C[0,1]$ with the following property:

For each k $(k = 0,1,2,\ldots)$, there is an $M_k > 0$, such that

$$|B_n(f_\alpha,k,a) - f_\alpha(a)| \geq M_k n^{-\frac{\alpha}{2}} .$$

In other words, no matter how smooth a function f is inside $[a,b]$, it is possible that $B_n(f,k,a)-f(a) \neq 0(n^{-(k+1)})$, or even not of $0(n^{-1})$ (even, still worse, $\neq 0(n^{-\beta})$ for any $\beta > 0$). Thus, the example justifies the condition in Theorem II.1.4 of shrinking the corresponding intervals.

Let $a = \frac{1}{2}$, $b > \frac{1}{2}$, and $\alpha \in (0,2)$. Let $f \in C[0,1]$ be defined as

$$f(x) = \begin{cases} (\frac{1}{2} - x)^{\alpha} & x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} .$$

Then we have

$$B_n(f,\tfrac{1}{2}) = \sum_{\nu=0}^{n} \binom{n}{\nu} (\tfrac{1}{2})^{\nu} (1-\tfrac{1}{2})^{n-\nu} f(\tfrac{\nu}{n})$$

$$= \frac{1}{2^n} \sum_{\nu \leq \frac{1}{2}} \binom{n}{\nu} (\tfrac{1}{2} - \tfrac{\nu}{n})^{\alpha} .$$

Further,

$$\sum_{\nu \leq \frac{1}{2} n} \binom{n}{\nu} (\tfrac{1}{2} - \tfrac{\nu}{n})^{\alpha} = \sum_{\nu \leq \frac{1}{2} n} \binom{n}{n-\nu} (\tfrac{n-\nu}{n} - \tfrac{1}{2})^{\alpha}$$

$$= \sum_{\nu \geq \frac{1}{2} n} \binom{n}{\nu} (\tfrac{\nu}{n} - \tfrac{1}{2})^{\alpha} .$$

Hence,

$$\sum_{\nu \leq \frac{1}{2} n} \binom{n}{\nu} (\tfrac{1}{2} - \tfrac{\nu}{n})^{\alpha} \doteq \frac{1}{2} \sum_{\nu} \binom{n}{\nu} \left| \frac{1}{2} - \frac{\nu}{n} \right|^{\alpha} .$$

In other words, $B_n(f, \frac{1}{2}) \doteq \frac{1}{2} B_n(|\frac{1}{2} - x|^{\alpha}, \frac{1}{2})$. Following ([6], pp. 565-566), we have

$$B_n(|\tfrac{1}{2} - x|^\alpha, \tfrac{1}{2}) = \int_o^1 W(n,t,u)|\tfrac{1}{2} - u|^\alpha \, du$$

$$= \frac{(2-2^{\frac{\alpha}{2}})+\epsilon_n}{\sqrt{\pi}} (2n)^{-\frac{\alpha}{2}} \int_0^\infty u^\alpha e^{-u^2} du + o(n^{-\frac{\alpha}{2}})$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Thus, by (1.10)

$$B_n(f,k,\tfrac{1}{2})-f(\tfrac{1}{2}) = \sum_j C(j,k)B_{2^j n}(f,\tfrac{1}{2})-f(\tfrac{1}{2})$$

$$= \sum_j C(j,k)\, 2^{-\frac{\alpha}{2}j} \frac{(2-2^{\frac{\alpha}{2}}) + \epsilon_{2^j n}}{\sqrt{\pi}} (2n)^{-\frac{\alpha}{2}} \int_0^\infty u^\alpha e^{-u^2} du$$

$$+ o(n^{-\frac{\alpha}{2}})$$

and so

$$|B_n(f,k,\tfrac{1}{2}) - f(\tfrac{1}{2})| \geq M n^{-\frac{\alpha}{2}} .$$

3. On the other hand, the following example shows that there exists a function f, having second derivative everywhere and uniform continuous outside any neighbourhood of a single fixed point $(\frac{1}{2})$, but $||B_n(f,x)-f(x)||_{C(\Omega)} \neq 0(\frac{1}{n})$, if Ω contains a neighbourhood of the point $\frac{1}{2}$.

Example II:

Let $f(x) \in C[o,1]$ be defined by

$$f(x) = \begin{cases} |x - \frac{1}{2}|^{\frac{7}{2}} \sin \dfrac{1}{|x - \frac{1}{2}|} & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases} .$$

It is easy to verify that $f''(\frac{1}{2})$ exists and equal to 0, and for $x \neq \frac{1}{2}$,

$$f''(x) = \frac{35}{4} |x - \tfrac{1}{2}|^{\frac{3}{2}} \sin \frac{1}{|x - \frac{1}{2}|} - 5|x - \tfrac{1}{2}|^{\frac{1}{2}} \cos \frac{1}{|x - \frac{1}{2}|} -$$
$$- |x - \tfrac{1}{2}|^{-\frac{1}{2}} \sin \frac{1}{|x - \frac{1}{2}|} .$$

Let Ω be, say, the closed interval $[\frac{1}{4}, \frac{3}{4}]$. Then we have

$$\overline{\lim_{n \to \infty}}\, n \, ||B_n(f,t) - f(t)||_{C[\frac{1}{4}, \frac{3}{4}]}$$

$$\geq \overline{\lim_{n \to \infty}}\, n |B_n(f, \tfrac{1}{2} + \frac{2}{(4m+1)\pi}) - f(\tfrac{1}{2} + \frac{2}{(4m+1)\pi})|$$

$$= |\frac{35}{4} [\frac{2}{(4m+1)\pi}]^{\frac{3}{2}} - [\frac{(4m+1)\pi}{2}]^{\frac{1}{2}} |$$

$$\geq \frac{6}{7} [\frac{(4m+1)\pi}{2}]^{\frac{1}{2}} \qquad \text{for } m = 1,2,\ldots .$$

That is,

$$\overline{\lim_{n \to \infty}}\, n \, ||B_n(f,t) - f(t)||_{C[\frac{1}{4}, \frac{3}{4}]} = \infty .$$

§2. The Extension of the Saturation Result to Other Bernstein-Type Operators

The combinations of Bernstein Polynomials discussed by Butzer [8] and by us in Section 1, form a new class of operators converging to the function with faster rate. One can apply the same combinations to many other operators and obtain new approximation processes with faster convergence properties.

For instance, we shall apply these combinations to operators of the form

$$S_\lambda^i(f,t) = \int_0^\infty W_i(\lambda,t,u)f(u)\,du \quad ,$$

with the kernels given in (2.19) below, and shall obtain similar saturation results for these new operators.

Explicitly, let

$$(2.19)\quad \left\{\begin{array}{l} S_\lambda^i(f,t) = \int_0^\infty W_i(\lambda,t,u)f(u)du, \quad i = 1,2,3,4, \\ \text{where} \\ W_1(\lambda,t,u) = e^{-t\lambda} \sum_{k=0}^\infty \frac{(t\lambda)^k}{k!} \delta(u - \frac{k}{\lambda}) \\ W_2(n,t,u) = \frac{1}{(n-1)!} (\frac{n}{t})^n e^{-nu/t} u^{n-1} \\ W_3(\lambda,t,u) = e^{-\lambda(t+u)} \left[\sum_{n=1}^\infty \frac{(\lambda^2 t)^n u^{n-1}}{n!\,(n-1)!} + \delta(u) \right] \\ W_4(\lambda,t,u) = \sqrt{\frac{\lambda}{2\pi}} \; e^{-\lambda(u-t)^2/2} \end{array}\right.$$

$S^1_\lambda(f,t)$ is commonly known as Szasz's operator. $S^i_\lambda(f,t)$, $i = 2,3,4$, are essentially due to Post-Widder, Phillips and Weierstree respectively. Some of the properties of these operators have recently been investigated in [21]. See also the classical text [29].

Similar to Section 1, we define the combinations for $S^i_\lambda(f,t)$ as follows:

$$\text{(2.20)} \qquad \begin{cases} S^i_\lambda(f,0,t) = S^i_\lambda(f,t) \\ \\ S^i_\lambda(f,k,t) = (2^k-1)^{-1}[2^k S^i_{2\lambda}(f,k-1,t) - S^i_\lambda(f,k-1,t)] \end{cases} .$$

Also, let

$$I_i(f,\lambda,k,a,b) = \lambda^{k+1} \| S^i_\lambda(f,k,\cdot) - f(\cdot) \|_{C[a,b]} .$$

Then we have the following result.

<u>Theorem II.2.1</u>

Let $f \in C[0,\infty)$, $|f(t)| \leq M e^{Nt}$ for some M and N, $0 < a < a_1 < b_1 < b < \infty$, and $\lambda_m = \lambda_o 2^m$, $m = 1,2,3,\ldots$. For any $i \in \{1,2,3,4\}$, the following implications hold: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), and (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6), where

(1) $I_i(f,\lambda_m,k,a,b) = 0(1)$, $\lambda_m \to \infty$;

(2) $f^{(2k+1)} \in A.C.(a,b)$ and $f^{(2k+2)} \in L_\infty[a,b]$;

(3) $I_i(f,\lambda,k,a_1,b_1) = 0(1)$, $\lambda \to \infty$;

(4) $I_i(f,\lambda_m,k,a,b) = o(1), \quad \lambda \to \infty;$

(5) $f \in C^{2k+2}(a,b)$ and $\sum_{m=k+1}^{2k+2} Q_i(m,k,t)f^{(m)}(t) = 0$ in (a,b),

where $Q_i(m,k,t)$ are polynomials depending on k and i;

(6) $I_i(f,n,k,a_1,b_1) = o(1), \quad n \to \infty.$

The basic ideas from the proof of Theorem II.1.2 will apply and we shall only discuss the points for which necessary changes have to be made.

§2.1 The Necessary Changes in the Proof of the Saturation Theorem

First of all one notices that, for the case $k = 0$, the saturation theorem has been proved only for Szasz operators in [54], while for the other three operators, there is no theorem in hand to support us as a starting point of the induction process. Therefore, we assume the theorem is true for $k-1$ for Szasz operators, while for the other three operators we do not use induction.

The essential difference between this theorem and the Theorem II.1.2, namely, the unbounded intervals, is modulated by the growth condition $|f(t)| \leq M e^{Nt}$. This condition, owing to the following lemma, assures us that the operators will converge.

Lemma II.2.2

For any $\delta > 0$, $\alpha > 0$, $m > 0$ fixed, there holds

(2.21) $$\int_{|n-t|\geq\delta} W_i(\lambda,t,n)e^{\alpha u}\, du = o(\lambda^{-m}), \quad \lambda \to \infty .$$

Lemmas corresponding to Lemmas II.1.3 and II.1.5 in the proof of Theorem II.1.2 are stated and proved similarly. In other words, we have the following:

Lemma II.2.3

Let f satisfy the conditions in the theorem and $\lambda_m = 2^m\lambda_o$, $m = 1,2,3,\ldots$. Then for each $i \in \{1,2,3,4\}$, the following holds:

(1) $\lambda_m^{k+1} \|S^i_{\lambda_m}(f,k,\cdot)-f(\cdot)\|_{C[a,b]} = O(1)$ is equivalent to $\lambda_m^{k+1}\|S^i_{2\lambda_m}(f,k,\cdot)-S^i_{\lambda_m}(f,k,\cdot)\|_{C[a,b]} = O(1)$;

(2) For any fixed $\delta > 0$, $\lambda_m^{k+\delta}\|S^i_{2\lambda_m}(f,k,\cdot) - S^i_{\lambda_m}(f,k,\cdot)\|_{C[a,b]} = O(1)$ implies

$$\lambda_m^k\|S^i_{2\lambda_m}(f,k-1,\cdot) - S^i_{\lambda_m}(f,k-1,\cdot)\|_{C[a,b]} = O(1) .$$

We remark that we shall not use Lemma II.2.3 (2) for the operators $i = 2,3,4$. However, we do need this lemma to conclude that, in the case of Szasz operators, $f^{(2k)} \in L_\infty[a,b]$ if f satisfies (1) of the theorem.

The lemma corresponding to Lemma II.1.4 is also similar while the proof is not as simple:

Lemma II.2.4

Let f satisfy the conditions in the theorem. If, in addition, $f^{(2k+2)}(t)$ exists, then

$$(2.22)\qquad \lambda^{k+1}(S^i_{2\lambda}(f,k,t)-S^i_\lambda(f,k,t))$$

$$= \sum_{j=k+1}^{2k+2} Q_i(j,k,t)f^{(j)}(t) + o(1)$$

$$\equiv P_{i,k}(D)f(t) + o(1) \quad ,$$

where $Q_i(j,k,t)$ are polynomials in t. Moreover,

$$Q_i(2k+2,k,t) = c_{i1}p_i(t)^{k+1}$$

$$Q_i(2k+1,k,t) = c_{i2}p_i(t)^k p_i'(t)$$

where

$$(2.23)\qquad p_i(t) = t, \quad p_2(t) = t^2, \quad p_3(t) = 2t, \quad p_4(t) = 1 \quad ,$$

and the $c_{i,j}$ are absolute constant.

If $f \in C^{2k+2}[a,b]$, then (2.22) is uniform in every interior interval $[a_1,b_1] \subset (a,b)$.

In the lemma corresponding to Lemma II.1.6, two changes are necessary. Firstly, we need to replace $||f|| \equiv \sup_t |f(t)|$, for in this case, the supremum is unbounded. Secondly, for the operators other than the Szasz operator, since we have not had $f^{(2k)} \in L_\infty[a,b]$ from the induction hypothesis, we cannot choose a sequence $\{f_\ell\}$ converging to f in the "$L^{2k}[a,b]$-norm" - a norm depending on 2k-derivatives of the function.

First, we define the modified norms as follows:

<u>Notations:</u> Let $C_N[0,\infty) = \{f \in C[0,\infty); |f(t)| \le Me^{Nt}\}$ and $L^{2k}_N[a,b] = \{f \in C_N[0,\infty);\ ^{2k} \in L_1[a,b]\}$. Define norms $||\cdot||_{C_N}$ and

$||\cdot||_{L_N^{2k}[a,b]}$ on $C_N[0,\infty)$ and $L_N^{2k}[a,b]$ respectively as follows:

$$||f||_{C_N} = \sup_{0\le t<\infty} |f(t)|e^{-Nt}$$

$$||f||_{L_N^{2k}[a,b]} = ||f||_{C_N} + \max_{0\le i\le 2k} ||f^{(i)}||_{L_1[a,b]} .$$

Now we can state the rectified lemma:

Lemma II.2.5

Let $g \in C_o^\infty$ with supp $g \subset (a,b)$. Then the following holds:

(1) If $f \in C_N[0,\infty)$, then for $i = 2,3,4$,

$$|\lambda^{k+1} <[S_{2\lambda}^i(f,k,.)-S_\lambda^i(f,k,\cdot)],g(\cdot)>| \le K_1||f||_{C_N} \quad ; \tag{2.24}$$

(2) If $f \in L_N^{2k}[a,b]$, then

$$|\lambda^{k+1} <[S_{2\lambda}^1(f,k,\cdot)-S_\lambda^1(f,k,\cdot)],g(\cdot)>| \le K_2||f||_{L_N^{2k}[a,b]} . \tag{2.25}$$

where K_1 and K_2 are constants depending only on g and its derivatives.

This lemma will justify the change of order of limits in

$$\lim_{\ell\to\infty} \lim_{\lambda_i\to\infty} \lambda_i^{k+1}[S_{2\lambda_i}(f_\ell,k,\cdot)-S_{\lambda_i}(f,k,\cdot),g(\cdot)> \tag{2.26}$$

$$= \lim_{\lambda_i\to\infty} \lim_{\ell\to\infty} \lambda_i^{k+1}[S_{2\lambda_i}(f_\ell,k,\cdot)-S_{\lambda_i}(f,k,\cdot),g(\cdot)>$$

where f_ℓ is a convergent sequence in $L_N^{2k}[a,b]$.

§2.2 The Asymptotic Result

In order to prove Lemma II.2.4, we begin with some preliminary lemmas.

Lemma II.2.6

For $i = 1,\ 2,$ and 4, there holds

$$(2.27) \qquad \frac{\partial}{\partial t} W_i(\lambda,t,u) = \frac{\lambda}{p_i(t)} W_i(\lambda,t,u)(u-t) \quad ,$$

where $p_i(t)$ are given in (2.23).

Proof.

The proof follows by direct calculation:

$$\frac{\partial}{\partial t} W_1(\lambda,t,u) = -\lambda W_1(\lambda,t,u) + e^{-t\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} (t\lambda)^k \frac{k}{\lambda} \frac{\lambda}{t} \delta(u - \frac{k}{\lambda})$$

$$= \frac{\lambda}{t} W_1(\lambda,t,u)(u-t)$$

$$\frac{\partial}{\partial t} W_2(n,t,u) = \frac{1}{(n-1)!} (\frac{n}{t})^n e^{-\frac{n}{t}u} u^{n-1}(-\frac{n}{t}) +$$

$$+ \frac{1}{(n-1)!} (\frac{n}{t})^n e^{-\frac{n}{t}u} u^{n-1}(\frac{n}{t^2} u)$$

$$= \frac{n}{t^2} W_2(n,t,u)(u-t) \quad .$$

$$\frac{\partial}{\partial t} W_4(\lambda,t,u) = \lambda W_4(\lambda,t,u)(u-t) \quad .$$

For the kernel $W_3(\lambda,t,u)$, relation (2.27) is not satisfied. This fact makes the investigation of $S_\lambda^3(f,k,t)$ difficult

but more interesting. We have the following lemma instead:

Lemma II.2.7

If $\mathbb{P} = t(1+\frac{D}{\lambda})^2 \equiv t(1+\frac{2}{\lambda}D+\frac{1}{\lambda^2}D^2)$, where $D = \frac{d}{dt}$,

then

$$\mathbb{P}\, W_3(\lambda,t,u) = W_3(\lambda,t,u)u.$$

Proof.

Beginning with the expression

$$e^{\lambda t}\, S_\lambda(f,t) = \int_0^\infty e^{-\lambda u}\left[\sum_{n=1}^\infty \frac{(\lambda^2 t)^n\, u^{n-1}}{n!(n-1)!} + \delta(u)\right] f(u)\, du \,,$$

we differentiate twice with respect to t on both sides to obtain

$$e^{\lambda t}(1+\lambda D)^2\, S_\lambda(f,t) = \frac{\lambda^2}{t}\int_0^\infty e^{\lambda t}\, W_3(\lambda,t,u)uf(u)du \quad .$$

Dividing both sides by $\frac{\lambda^2}{t}\, e^{\lambda t}$ yields the required relation.

Remark

In order to calculate $S_\lambda^3(x^i,t)$, one should simply computes $\mathbb{P}^i \cdot 1$. We show some examples in the following table.

$$\mathbb{P} \equiv t\left(1 + \frac{2}{\lambda} D + \frac{1}{\lambda^2} D^2\right)$$

$$\mathbb{P}^0 \; 1 = 1$$

$$\mathbb{P}^1 \; 1 = t$$

$$\mathbb{P}^2 \; 1 = t^2 + \frac{2}{\lambda} t$$

$$\mathbb{P}^3 \; 1 = t^3 + 6 \frac{t^2}{\lambda} + 6 \frac{t}{\lambda^2}$$

$$\mathbb{P}^4 \; 1 = t^4 + 12 \frac{t^3}{\lambda} + 36 \frac{t^2}{\lambda^2} + 24 \frac{t}{\lambda^3}$$

$$\mathbb{P}^5 \; 1 = t^5 + 20 \frac{t^4}{\lambda} + 120 \frac{t^3}{\lambda^2} + 240 \frac{t^2}{\lambda^3} + 120 \frac{t}{\lambda^2}$$

$$\mathbb{P}^6 \; 1 = t^6 + 30 \frac{t^6}{\lambda} + 300 \frac{t^4}{\lambda^2} + 1200 \frac{t^3}{\lambda^3} + 1800 \frac{t^2}{\lambda^4} + 720 \frac{t}{\lambda^5}$$

$$\mathbb{P}^7 \; 1 = t^7 + 42 \frac{t^6}{\lambda} + 630 \frac{t^5}{\lambda^2} + 4200 \frac{t^4}{\lambda^3} + 12600 \frac{t^3}{\lambda^4} + 15120 \frac{t^2}{\lambda^5} + 5040 \frac{t^1}{\lambda^6}$$

$$\mathbb{P}^8 \; 1 = t^8 + 56 \frac{t^7}{\lambda} + 1176 \frac{t^6}{\lambda^2} + 11760 \frac{t^5}{\lambda^3} + 58800 \frac{t^4}{\lambda^4} + 141120 \frac{t^3}{\lambda^5}$$
$$+ 141120 \frac{t^2}{\lambda^6} + 40320 \frac{t}{\lambda^7}$$

$$\mathbb{P}^9 \; 1 = t^9 + 72 \frac{t^8}{\lambda} + 2016 \frac{t^7}{\lambda^2} + 28224 \frac{t^6}{\lambda^3} + 211680 \frac{t^5}{\lambda^4} + 846720 \frac{t^4}{\lambda^5}$$
$$+ 1693440 \frac{t^3}{\lambda^6} + 1451520 \frac{t^2}{\lambda^7} + 362880 \frac{t}{\lambda^8}$$

$$\mathbb{P}^{10} \; 1 = t^{10} + 90 \frac{t^9}{\lambda} + 3240 \frac{t^8}{\lambda^2} + 60480 \frac{t^7}{\lambda^3} + 635040 \frac{t^6}{\lambda^4} + 3810240 \frac{t^5}{\lambda^5}$$
$$+ 12700800 \frac{t^4}{\lambda^6} + 21772800 \frac{t^3}{\lambda^7} + 16329600 \frac{t^2}{\lambda^8} + 3628800 \frac{t}{\lambda^9}$$

$$\mathbb{P}^{11} \; 1 = t^{11} + 110 \frac{t^{10}}{\lambda} + 4950 \frac{t^9}{\lambda^2} + 118800 \frac{t^8}{\lambda^3} + 1663200 \frac{t^7}{\lambda^4}$$
$$+ 13970880 \frac{t^6}{\lambda^5} + 69854400 \frac{t^5}{\lambda^6} + 199584000 \frac{t^4}{\lambda^7}$$
$$+ 299376000 \frac{t^3}{\lambda^8} + 199584000 \frac{t^2}{\lambda^4} + 39916800 \frac{t}{\lambda^{10}}$$

$$\mathbb{P}^{12}\, 1 = t^{12} + 132\,\frac{t^{11}}{\lambda} + 7260\,\frac{t^{10}}{\lambda^{2}} + 217800\,\frac{t^{9}}{\lambda^{3}} + 3920400\,\frac{t^{8}}{\lambda^{4}}$$

$$+ 43908480\,\frac{t^{7}}{\lambda^{5}} + 307359360\,\frac{t^{6}}{\lambda^{6}} + 1317254400\,\frac{t^{5}}{\lambda^{7}}$$

$$+ 3293136000\,\frac{t^{4}}{\lambda^{8}} + 4390848000\,\frac{t^{3}}{\lambda^{4}} + 2634508800\,\frac{t^{2}}{\lambda^{10}}$$

$$+ 479001600\,\frac{t}{\lambda^{11}}\ .$$

<u>Lemma II.2.8</u>

Let $A^{i}_{m}(\lambda,t)$ be given by

$$A^{i}_{m}(\lambda,t) = \lambda^{m} \int_{0}^{\infty} W_{i}(\lambda,t,u)(u-t)^{m}\, du \qquad m = 0,1,2,\ldots\ . \tag{2.28}$$

Then for each i, $i = 1,2,3,4$, the following statements hold:

(1) $A^{i}_{m}(\lambda,t)$ is a polynomial in λ and t;

(2) the degree of $A^{i}_{m}(\lambda,t)$ in λ is $[\frac{m}{2}]$;

(3) the coefficient of λ^{m} in the polynomial $A^{i}_{2m}(\lambda,t)$ is $c_1\, p_i(t)^{m}$ and in the polynomial $A^{i}_{2m+1}(\lambda,t)$ is $c_2\, p^{m}_{i}(t)p'_{i}(t)$; where $p_i(t)$ are given in equation (2.23).

<u>Proof.</u>

For $i \neq 3$, we can easily deduce from Lemma II.2.6 the recursion relation:

$$A_m^i(\lambda,t) = m\lambda\, p_i(t)\, A_{m+1}^i(\lambda,t) + p_i(t)\,\frac{d}{dt} A_m^i(\lambda,t) \quad ,$$

and the rest follows easily by induction.

For $i = 3$, we first prove an intermediate result:

<u>Assertion</u>:

If $\mathbb{P} \equiv t(1+\frac{D}{\lambda})^2$, $f,g \in C^\infty$, then

$$\mathbb{P}(f\cdot g) = (\mathbb{P}f)g + \frac{2}{\lambda}\, tf(Dg) - \frac{t}{\lambda^2}\, f\cdot(D^2 g) + \frac{t}{\lambda^2}\, D[fDg] \ .$$

<u>Proof of the Assertion</u>:

Observe that

$$\mathbb{P}(f\cdot g) = t(1+\frac{2}{\lambda}D + \frac{D^2}{\lambda^2})fg = t\{fg + \frac{2}{\lambda}(Df)g + \frac{1}{\lambda^2}(D^2 f)g +$$

$$+ \frac{2}{\lambda}\, fDg + \frac{1}{\lambda^2}\, fD^2 g + \frac{2}{\lambda^2}(Df)(Dg)\} \ .$$

Substituting $(Df)(Dg)$ for $D(fDg) - f(D^2 g)$, we obtain the relation.

We use induction and the above assertion to prove the lemma for $i = 3$.

The lemma is trivial for $m = 0,1,2$. Assume that the lemma is true for integers less than or equal to m. Applying the operator $(\frac{2t}{\lambda}D + \frac{t}{\lambda^2}D^2) = \mathbb{P} - t$ to both sides of (2.9), we have

$$\frac{2t}{\lambda}\, D\, A_m^3(\lambda,t) + \frac{t}{\lambda^2}\, D^2\, A_m^3(\lambda,t)$$

$$= \mathbb{P}\,\lambda^m \int_0^\infty W_3(\lambda,t,u)(u-t)^m du - t\,\lambda^m \int_0^\infty W_3(\lambda,t,u)(u-t)^m\, du$$

$$= \lambda^m \left\{ \int_0^\infty [\, PW_3(\lambda,t,u)](u-t)^m du - t\lambda^m \int_0^\infty W_3(\lambda,t,u)(u-t)^m du \right\}$$

$$+ 2t\lambda^{m-1} \int_0^\infty W_3(\lambda,t,u)D(u-t)^m \, du$$

$$- t\, \lambda^{m-2} \int_0^\infty W_3(\lambda,t,u)D^2(u-t)^m du$$

$$+ 2t\, \lambda^{m-2}\, D \int_0^\infty W_3(\lambda,t,u)D(u-t)^m du$$

$$= \frac{1}{\lambda} A^3_{m+1}(\lambda,t) - 2tmA^3_{m-1}(\lambda,t) - tm(m-1)A^3_{m-2}(\lambda,t)$$

$$- \frac{2tm}{\lambda} D\, A^3_{m-1}(\lambda,t) \; .$$

Thus, for $i = 3$, the lemma is proved by induction.

Lemma II.2.4 now follows easily by the above lemma, the analogue of Lemma II.1.9, and the appropriate modifications of the proof of Lemma II.1.4, the details of which we shall omit.

§2.3 Justification of the Interchange of the Order of Limits: Proof of Lemma II.2.5

The proof of this lemma which allows us to interchange the limits in (2.26) is quite involved so we shall divide the proof into several steps. The first five steps will prove part (1) of the lemma, while the remaining step deals with part (2).

1° We first get the integration over a finite interval, and break the integration into several parts which are easier to estimate.

For $g \in C_o^\infty$, $\operatorname{supp} g \subset (a,b)$, and by using the analogue to equation (2.10), we have

$$\lambda^{k+1} <[S^i_{2\lambda}(f,k,\cdot) - S^i_\lambda(f,k,\cdot)],g(\cdot)>$$

$$= \lambda^{k+1} \int_0^\infty \int_0^\infty \left\{ \sum_{j=0}^{k+1} \alpha(j,k) W_i(2^j\lambda,t,u) f(u) g(t) \right\} du dt$$

$$= \lambda^{k+1} \int_{\operatorname{supp} g} \int_0^\infty \left\{ \sum_{j=0}^{k+1} \alpha(j,k) W_i(2^j\lambda,t,u) f(u) g(t) \right\} du\, dt$$

$$= \lambda^{k+1} \int_{\operatorname{supp} g} \int_a^b \{ \ldots \}\, du\, dt + o(1) ||f||_{C_N}$$

$$= \lambda^{k+1} \int_0^\infty \int_a^b \{ \ldots \}\, du\, dt + o(1) ||f||_{C_N} .$$

By Fubini's Theorem, this expression can be rewritten as

$$\lambda^{k+1} \int_a^b \int_0^\infty \{ \ldots \}\, dt\, du + o(1) ||f||_{C_N}$$

$$= \lambda^{k+1} \int_a^b \int_0^\infty \sum_{\gamma=0}^{2k+2} \sum_{j=0}^{k+1} \frac{1}{\gamma!} \alpha(j,k) W_i(2^j\lambda,t,u) f(u) \cdot$$

$$\cdot\, g^{(\gamma)}(u)(t-u)^\gamma dt\, du$$

$$+ \lambda^{k+1} \int_a^b \int_0^\infty \sum_{j=0}^{k+1} \alpha(j,k) W_i(2^j\lambda,t,u) f(u) \varepsilon(t,u)(t-u)^{2k+2} dt du$$

$$+ o(1) ||f||_{C_N}$$

$$= \sum_{\gamma=0}^{2k+2} \lambda^{k+1} \int_a^b \int_0^\infty \sum_{j=0}^{k+1} \frac{1}{\gamma!} \alpha(j,k) W_i(2^j\lambda,t,u) f(u) \cdot$$

$$\cdot\, g^{(\gamma)}(u)(t-u)^\gamma dt\, du$$

$$+\ \lambda^{k+1}\int_a^b\int_0^\infty\sum_{j=0}^{k+1}\alpha(j,k)W_i(2^j\lambda,t,u)f(u)\varepsilon(t,u)(t-u)^{2k+2}dtdu$$

$$+\ o(1)\|f\|_{C_N}$$

$$\equiv\sum_{\gamma=0}^{2k+3}M_\gamma^{(i)}+o(1)\|f\|_{C_N}\ .$$

On the other hand, this also may be rewritten as

$$\sum_{\gamma=0}^{2k+2}\lambda^{k+1}\int_a^b\int_0^\infty\sum_{j=0}^{k+1}\alpha(j,k)W_i(2^j\lambda,f,u)\psi_\gamma(u)t^\gamma dt\ du$$

$$+\ \lambda^{k+1}\int_a^b\int_0^\infty\sum_{j=0}^{k+1}\alpha(j,k)W_i(2^j\lambda,t,u)f(u)\varepsilon(t,u)(t-u)^{2k+2}dtdu$$

$$+\ o(1)\|f\|_{C_N}$$

$$\equiv\sum_{\gamma=0}^{2k+3}I_\gamma(i)+o(1)\|f\|_{C_N}\ ,$$

where

$$\psi_\gamma(u)=\sum_{m=\gamma}^{2k+2}(-1)^{m-\gamma}\frac{1}{m!}f(u)g^{(m)}(u)u^{m-\gamma}\ .$$

(Note that $M_{2k+3}^{(i)}\equiv I_{2k+3}^{(i)}$.)

2° In this step, we estimate $I_{2k+3}(i)$ $(=M_{2k+3}(i))$ for $i=1,3,4$.

First, notice that

$$|\varepsilon(t,u)| = \frac{2}{(2k+2)!} \; |g^{(2k+2)}(\xi)|$$

$$\leq \frac{2}{(2k+2)!} \, ||g^{(2k+2)}||_{C[a,b]} < \infty$$

and

$$|f(u)| \leq ||f||_{C[a,b]} \leq e^{Nb} ||f||_{C_N} \quad .$$

It follows that

$$|I_{2k+3}(i)| \leq M||f||_{C_N} \; \lambda^{k+1} \int_0^\infty \int_a^b \sum_{j=0}^{k+1} |\alpha(j,k)| \cdot$$

$$\cdot \; W_i(2^j\lambda,t,u) \cdot (t-u)^{2(k+1)} du \; dt \quad ,$$

For $i = 1$, 3, and 4, $p_i(t)$ is a linear polynomial in t. In such cases, $I_{2k+3}(i)$ can be estimated as follows:

$$\lambda^{k+1} \int_0^\infty \int_a^b W_i(\lambda,t,u)(t-u)^{2k+2} \; du \; dt$$

$$= \lambda^{k+1} \int_0^{b+1} \int_a^b W_i(\lambda,t,u)(t-u)^{2k+2} \; du \; dt$$

$$+ \lambda^{k+1} \int_{b+1}^\infty \int_a^b W_i(\lambda,t,u)(t-u)^{2k+2} \; du \; dt$$

$$\equiv J_1 + J_2 \quad .$$

Further, $J_1 \leq M$ by Lemma II.2.8, since t is bounded, and, again by Lemma II.2.8,

$$J_2 \leq \lambda^{k+1} \int_{b+1}^{\infty} \int_{a}^{b} W_i(\lambda,t,u)(t-u)^{2k+2}\left(\frac{t-u}{t-b}\right)^{2m} du dt$$

$$\leq M \int_{b+1}^{\infty} \frac{p_i(t)^{k+m+1}}{(t-b)^{2m}} du \quad .$$

By choosing $m \geq k+3$, we arrive at $J_2 \leq M'$.

3° For estimating $I_{2k+3}(2)$, we need the following asymptotic relation:

<u>Lemma II.2.9</u>

If

(2.29) $$F_n(k,c) \equiv n^k \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} \frac{n^{j+1}(n-c-j-1)!}{(n-c)!}$$

$$\equiv n^k \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} \frac{n^{j+1}}{(n-c)(n-c-1)\ \dots\ (n-c-j)} ,$$

then, for any fixed positive integers k and c, the asymptotic relation

$$F_n(k,c) = 0(1) \qquad \text{as} \qquad n \to \infty$$

holds.

The required relation, namely, $I_{2k+3}(2) = 0(1)$, is a direct consequence of the above assertion. In fact

$$n^{k+1} \int_{a}^{b} \int_{0}^{\infty} W_2(n,t,u)(u-t)^{2k+2} dt\ du$$

$$= n^{k+1} \int_{a}^{b} \sum_{j=0}^{2k+1} \binom{2k+2}{j} (-1)^j u^{2k+2-j} \left[\frac{n^{j+1}(n-2-j)!}{(n-1)!} u^j\right] du$$

$$= \{n^{k+1} \sum_{j=0}^{2k+2} \binom{2k+2}{j} (-1)^j \frac{n^{j+1}(n-2-j)!}{(n-1)!}\} \int_a^b u^{2k+2}\, du$$

$$= F_n(k+1,1) \left[\int_a^b u^{2k+2}\, du \right].$$

Proof of the Lemma

The proof uses induction on k, for any positive integers c. Direct calculation shows

$$F_n(1,c) = n \left\{ \frac{1}{1-\frac{c}{n}} - \frac{2}{(1-\frac{c}{n})(1-\frac{c+1}{n})} + \frac{1}{(1-\frac{c}{n})(1-\frac{c+1}{n})(1-\frac{c+2}{n})} \right\}$$

$$= \frac{1}{(1-\frac{c}{n})(1-\frac{c+1}{n})} \left[\frac{c+2}{(1-\frac{c+2}{n})} - (c+1) \right]$$

$$= O(1) .$$

Now assume $F_n(k,c) = O(1)$ for any $c \in \{1,2,\ldots\}$, as $n \to \infty$. Since $\binom{2k+2}{j+1} - \binom{2k+1}{j} = \binom{2k+1}{j+1}$, we can combine j^{th} term with $(j-1)^{th}$ term and $(j+1)^{th}$ term, $j = 1,2,\ldots,2k+1$, in

$$F_n(k+1,c) = n^{k+1} \sum_{j=0}^{2k+2} (-1)^j \binom{2k+2}{j} \frac{1}{(1-\frac{c}{n})(1-\frac{c+1}{n})\ldots(1-\frac{c+j}{n})}$$

to obtain

$$F_n(k+1,c) = n^{k+1} \sum_{j=0}^{2k+1} (-1)^{j+1} \frac{\frac{c+j+1}{n}\binom{2k+1}{j}}{(1-\frac{c}{n})\ \ldots\ (1-\frac{c+j+1}{n})}$$

Notice that

$$(c+j+1)\binom{2k+1}{j} = (c+j+1)\,\frac{(2k+1)!}{j!(2k+1-j)!}$$

$$= \frac{(2k)!}{j!(2k+1-j)!}\{(c+1)(2k-j+1) + (c+2k+2)j\}$$

$$= \frac{(c+1)(2k)!}{j!(2k-j)!} + \frac{(c+2k+2)(2k)!}{(j-1)!(2k+1-j)!}$$

$$= (c+1)\binom{2k}{j} + (c+2k+2)\binom{2k}{j-1}$$

Therefore, we have

$$F_n(k+1,c) = n^k \left\{ (c+2k+2) \sum_{j=0}^{2k} \frac{(-1)^j \binom{2k}{j}}{(1-\frac{c}{n}) \cdots (1-\frac{c+j+2}{n})} - (c+1) \sum_{j=0}^{2k} \frac{(-1)^j \binom{2k}{j}}{(1-\frac{c}{n}) \cdots (1-\frac{c+j+1}{n})} \right\}$$

$$= \frac{c+2k+2}{(1-\frac{c}{n})(1-\frac{c+1}{n})} F_n(k,c+2) - \frac{c+1}{1-\frac{c}{n}} F_n(k,c+1)$$

$$= O(1) \,, \qquad \text{as} \quad n \to \infty .$$

Thus, the proof of the lemma is complete.

This completes the estimation in Step 3.

4° It remains to estimate $I_\gamma(i)$ or $M_\gamma(i)$, $\gamma \le 2k+2$. For $i = 4$, it would be more convenient to consider $M_\gamma(4)$. In this case, we look at the typical term

$$M_\gamma(4) = \lambda^{k+1} \int_a^b \frac{1}{\gamma!} \left[\sum_{j=0}^{k+1} \alpha(j,k) \int_0^\infty W_4(2^j\lambda,t,u)(u-t)^\gamma dt \right] \cdot$$

$$\cdot\ f(u) g^{(\gamma)}(u)\ du \quad .$$

By the symmetry of $W_4(\lambda,t,u)$ in t and u, Lemma II.2.8, and Lemma II.1.9,

$$M_\gamma(4) = \lambda^{k+1} \int_a^b \frac{1}{\gamma!} \left[\sum_{j=0}^{k+1} \alpha(j,k) \frac{A^4_\gamma(2^j\lambda,u)}{(2^j\lambda)^\gamma} \right] f(u) g(u)^\gamma du$$

$$= \begin{cases} 0 & \gamma = 0,1,\ldots,k \\ 0(1)\|f\|_{C_N} & \gamma = k+1,\ldots,2k+2 \end{cases} .$$

5° The estimate for $I_\gamma(i)$, $i = 2,3$, will require the following lemma.

<u>Lemma II.2.10</u>

Let $u \in [a,b] \subset (0,\infty)$, $j = 0,1,2,\ldots,2k+2$, and $i = 2,3$. Then

(1) $A^i_j(u) \equiv \int_0^\infty W_i(\lambda,t,u)t^j\, dt$ is a polynomial in u and $\frac{1}{\lambda}$ plus $o(\lambda^{-(k+1)})$;

(2) The degree of $A^i_j(u)$ in u is j ;

(3) The coefficient of u^j in $A^i_j(u)$ is 1.

Proof:

For $i = 2$, the lemma follows from the relation:

$$\int_0^\infty W_2(n,t,u)t^j \, dt = \frac{n^{j+1}\,(n-2-j)!}{(n-1)!}\, u^j \quad ,$$

and the Taylor expansion of $(1/(1-\frac{i}{n}))$, $i = 1,2,\ldots,j+1$.

To prove the case for $i = 3$, we shall use techniques similar to those used in Lemma II.2.8 for this case.

First of all, direct checking shows that the lemma is true for $j = 0,1,2$. In fact, we have

$$\int_0^\infty W_3(\lambda,t,u)t^j \, dt = \begin{cases} 1 & j = 0 \ , \\ u + \frac{2}{\lambda} & j = 1 \ , \\ u^2 + \frac{6}{\lambda}u + \frac{6}{\lambda^2} & j = 2 \ . \end{cases}$$

We shall obtain a recursion relation that will allow the induction. First, we integrate $A_j^3(u)$ by parts two times to obtain

$$\begin{aligned} A_j^3(u) &= \int_0^\infty W_3(\lambda,t,u)t^j \, dt \\ &= -\frac{1}{j+1}\int_0^\infty (DW_3(\lambda,t,u))t^{j+1} \, dt \\ &= \frac{1}{(j+1)(j+2)}\int_0^\infty (D^2W_3(\lambda,t,u))t^{j+2} \, dt \ . \end{aligned}$$

It follows that

$$\int_0^\infty [\,\mathbb{P}W_3(\lambda,t,u)]t^j\,dt - \int_0^\infty W_3(\lambda,t,u)t^{j+1}\,dt$$

$$= \int_0^\infty [(\frac{D^2}{\lambda^2} + \frac{2}{\lambda}D)W_3(\lambda,t,u)]t^{j+1}\,dt$$

$$= \frac{j(j+1)}{\lambda^2} A^3_{j-1}(u) - \frac{2(j+1)}{\lambda} A^3_j(u) \quad .$$

Since $\mathbb{P}W_3(\lambda,t,u) = W_3(\lambda,t,u)u$, the above relation implies that

$$A^3_{j+1}(u) = uA^3_j(u) + \frac{2(j+1)}{\lambda} A^3_j(u) = \frac{j(j+1)}{\lambda^2} A^3_{j-1}(u) \quad .$$

The lemma readily follows by induction.

The estimation for $I_\gamma(i)$, $i = 2,3$, follows from the application of the above lemma and Lemma II.1.9.

6° We shall deal with the Szasz operator $(i = 1)$ in this step. We emphasize that this is the only step we assume $f \in L^{2k}_N[a,b]$ by the induction hypothesis.

Since it is obvious that, for $f \in L^{2k}_N[a,b]$, $||f||_{C_N} \leq ||f||_{L^{2k}_N[a,b]}$, we can use Steps 1° and 2° for this operator by changing $||f||_{C_N}$ to $||f||_{L^{2k}_N[a,b]}$.

The estimate of $I_\gamma(1)$ for $\gamma \leq 2k+2$ is similar to the proof of Lemma II.1.6 for the case of Bernstein polynomials. This estimate also involves the use of the Euler-McLaurin formula and

details will be omitted.

The proof of Lemma II.2.5, and hence of Theorem II.2.1, is complete.

CHAPTER III

GENERALIZATIONS

§1. Saturation Result for Baskakov Operators

We have seen in the last chapter that a number of operators have similar convergence properties. A natural question arises: Does a similar result hold for some general operator?

In a private converation, Professor G.G. Lorentz inquired as to the possibility of obtaining a similar saturation result for some "general" operators, for instance the Baskakov operators. The saturation results for these operators are resolved in this chapter.

The definition of Baskakov operators given below can be found in [54].

<u>Definition III.1.1</u>

The family of operators $\{M_\lambda\}$ are Baskakov operators if

(3.1) $$M_\lambda(f,x) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi_\lambda^{(k)}(x)}{k!} x^k f(\frac{k}{\lambda}) ,$$

where $\{\phi_\lambda\}$ is a family of real-valued functions having the following properties:

(1) $\phi_\lambda(x)$ can be expanded in Taylor's series in $[0,\beta)$ (β may be equal to ∞);

(2) $\phi_\lambda(0) = 1$;

(3) $(-1)^k \phi_\lambda^{(k)}(x) \geq 0$ $(k = 0,1,2,\ldots)$ for $x \in [0,\beta)$;

(4) $-\phi_\lambda^{(k)}(x) = \lambda \phi_{\lambda+c}^{(k-1)}(x)$ $(k = 1,2,\ldots)$, $x \in [0,\beta)$, for

some constant c;

(5) For any fixed constant M, $\lim_{x \to \infty} \phi_\lambda(x)x^k = 0$ for $k = 0,1,2,\ldots,M$.

Examples of Baskakov operators are:

(a) $\phi_n(x) = (1-x)^n$, $(c = -1)$,

$$B_n(f,x) = \sum_{k=0}^{n} f(\tfrac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} \quad ;$$

(b) $\phi_n(x) = e^{-nx}$, $(c = 0)$,

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty} f(\tfrac{k}{n})\frac{n^k x^k}{k!} \quad ;$$

(c) $\phi_n(x) = (1+x)^{-n}$, $(c = 1)$,

$$L_n(f,x) = (1+x)^{-n}\sum_{k=0}^{\infty} f(\tfrac{k}{n})\frac{n(n+1)\ \ldots\ (n+k-1)}{k!}\left(\frac{x}{1+x}\right)^k \; ;$$

(d) $\phi_\lambda(x) = (1+cx)^{-\frac{\lambda}{c}}$, $(c \neq 0)$.

In particular, (a) and (b) show that Baskakov operators include the Bernstein polynomials and Szasz operators respectively as special cases; on the other hand, (a) and (c) are special cases of (d).

We shall investigate the saturation problem for the same combinations as were investigated in the previous chapter. Explicitly, we define the following notation by induction:

(3.2) $$M_\lambda(f,k,t) = (2^k-1)^{-1}[2^k M_{2\lambda}(f,k-1,t)-M_\lambda(f,k-1,t)],$$

$$M_\lambda(f,0,t) = M_\lambda(f,t) \quad .$$

In [54] Suzuki has investigated only the case of continuous functions with compact supports (c.f., also [3]). The saturation result he achieved for Baskakov operators can actually be extended to functions with growth not faster than some polynomials, that is, to functions satisfying $|f(t)| \leq M(1+t)^A$ for some M and $A > 0$ (see Lemma III.1.3, below). This refined version will serve as the starting point for the proof of the following theorem.

<u>Theorem III.1.2</u>

Let $f \in C[0,\beta)$ and $|f(t)| \leq M(1+t)^A$ for some M and $A > 0$. If $\lambda_i = \lambda_o 2^i$ and $I(f,\lambda, k,a,b) = \lambda^{k+1} \|M (f,k,\cdot)-f(\cdot)\|_{C[a,b]}$, then, in the following, (1) ==> (2) ==> (3) and (4) ==> (5) ==>(6).

(1) $I(f,\lambda_i,k,a,b) = O(1)$;

(2) $f^{(2k+1)} \in A\cdot C\cdot(a,b)$ and $f^{(2k+2)} \in L_\infty[a,b]$;

(3) $I(f,\lambda,k,a_1,b_1) = O(1)$;

(4) $I(f,\lambda_i,k,a,b) = o(1)$;

(5) $f^{(2k+2)} \in C(a,b)$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t)f^{(i)}(t) = 0$

where $Q(i,k,f)$ are polynomials in t that depend on k;

(6) $I(f,\lambda,k,a_1,b_1) = o(1)$.

The proof is similar to the proof of Theorem II.2.1. Therefore, we shall not go through the details of the proof but shall state and prove the necessary changes.

For convenience, let $\psi_A(t) = (1+t)^A$ be the dominating function in the statement of the theorem. Also, let $W(\lambda,t,u) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi_\lambda^{(k)}(t)}{k!} t^k \delta(u - \frac{k}{\lambda})$ be the kernel that gives rise to the Baskakov operator $M_\lambda(f;t)$.

§1.1 Descriptions of the Changes in the Proof.

First of all, in order to ensure that $M_\lambda(f,k,t)$ is really an approximation process of functions f that are dominated by $\psi_A(t)$ for some $A > 0$, and that a similar saturation theorem to the one obtained in [54] remains true for such functions, we shall prove the following lemma for general positive linear approximation processes. The lemmas of this section will be proved in later sections.

Lemma III.1.3

Let $S_\lambda^*(f,t) = \int_\Omega W^*(\lambda,t,u)f(u)du$, $\lambda \in \Lambda$ be a positive linear approximation process such that

(1) $S_\lambda^*(1,t) = 1$,

(2) there exists an m, such that
$S_\lambda^*((x-t)^{2m},t) = O(\phi_\lambda(t))$, where $\phi_\lambda(t) \to 0$ as $\lambda \to \infty$ uniformly in any bounded interval.

Moreover, assume a function $g \geq 0$ is given such that $S_\lambda^*(g^2,t)$ is uniformly bounded in any bounded interval. Then if any of the following

(i) $S_\lambda^*(f,t) \to f(t)$, $\lambda \to \infty$ pointwise (or uniformly) for $t \in [a,b]$;

(ii) the condition $||S^*_{\lambda_n}(f,t)-f(t)||_{C[a,b]} = O(\sigma(\lambda_n))$ (or $o(\sigma(\lambda_n))$) would imply $f|_{[a,b]}$ belongs to a class $L[a,b]$, where

$$||\phi(t)||^{\frac{1}{2}}_{L_\infty[a,b]} = o(\sigma(\lambda)) ;$$

or

(iii) the condition $f|_{[a,b]} \in L[a,b]$ would imply

$$||S^*_\lambda(f,t)-f(t)||_{C[a,b]} = O(\sigma(\lambda)) \quad (\text{or } o(\sigma(\lambda))) ,$$

where $[a',b'] \subset [a,b]$ and $||\phi_\lambda(t)||_{L_\infty[a,b]} = o(\sigma(\lambda))$,

is true for functions having compact supports, then the corresponding result is also true for functions $|f(t)| \leq Mg(t)$.

Remarks:

(1) $g(t)$ can be chosen at least as $|1+t|^m$.

(2) The condition (2) can be replaced by

$$\int_{|u-t|\geq\delta} W^*(\lambda,t,u)\,du = O(\phi_\lambda(t)) .$$

For the above lemma, we need the following lemma.

Lemma III.1.4

For any positive integer m, the Baskakov operator satisfies

$$(3.3) \qquad M_\lambda((x-t)^{2m},t) = \begin{cases} 1 & m = 0 \\ O(\lambda^{-m}) & m > 0 \end{cases}$$

uniformly in any bounded interval.

If we denote the kernel of $M(f,t)$ by $W(\lambda,t,u)$. Then the above lemma can be re-represented as

$$\int_0^\beta W(\lambda,t,u)(u-t)^{2m}du = O(\lambda^{-m}) \quad .$$

Following the outline of the proof given in Chapter II, Section 1.1, we again reduce the saturation result for $\lambda^{k+1}\|M_\lambda(f,k,\cdot)-f(\cdot)\| = O(1)$ (or $o(1)$) to that of $\lambda^{k+1}\|M_{2\lambda}(f,k,\cdot) - M_\lambda(f,k,\cdot)\| = O(1)$ (or $o(1)$) in a completely analogous way.

The proof proceeds without change until we need an asymptotic result similar to Lemma II.1.4, the proof of which is more complicated.

<u>Lemma III.1.5</u>

Let f satisfy the condition of the theorem. If, in addition, $f^{(2k+2)}(t)$ exists, then

$$\lambda^{k+1}[M_{2\lambda}(f,k,t) - M_\lambda(f,k,t)] = \sum_{j=k+1}^{2k+2} Q(j,k,t)f^{(j)}(t) + o(1) \quad , \tag{3.4}$$

where $Q(j,k,t)$ are polynomials in t. Moreover,

$$Q(2k+2,k,t) = c_1\, p(t)^{k+1} \quad ,$$

$$Q(2k+1,k,t) = c_2(1+2ct)p(t)^k \quad ,$$

where $p(t) = t(1+ct)$, c_1, c_2 are constants, and c is the constant of condition (4) of Definition III.1.1.

If $f \in C^{2k+2}[a,b]$, then (3.4) is uniform in every interior interval $[a_1,b_1] \subset (a,b)$.

Proceeding through steps IV and V of the proof in Chapter II, Section 1.1, we only need the analog to Lemma II.1.5 for the operator $M_{2\lambda_i}(f,k,\cdot) - M_{\lambda_i}(f,k,\cdot)$ and for functions satisfying $f \in C[0,\beta)$ and $|f(t)| \leq M(1+t)^A$ for some M and $A > 0$. The proof is entirely similar so we shall not elaborate.

However, the crucial lemma which allows the interchange of limits in the analog to equation (2.5) does require further explanation.

Lemma III.1.6

Let $f \in C[0,\beta)$ and $f(x) \leq M\psi_A(x)$. If $f^{(2k)} \in L_\infty[a,b]$, $g \in C_o^\infty$, supp $g \subset (a,b)$, then

$$|\lambda^{k+1} < M_{2\lambda}(f,k,\cdot) - M_\lambda(f,k,\cdot)], g(\cdot)>|$$

$$\leq K \, ||f||_{(A)}$$

where

$$||f||_{(A)} = \sup_{0\leq t<\infty} |f(t)|\psi_A^{-1}(t) + \max_{0\leq i\leq 2k} ||f^{(i)}||_{L_1[a,b]}$$

and K is a constant depending only on g and its derivatives.

The rest of the proof proceeds as in Chapter II, Section 1.1. In the remainder of this chapter we shall prove the above lemmas.

§1.2 Some Preliminary Results

In the following lemma, we shall study the various properties of the kernel $W(\lambda,t,u)$ of $M_\lambda(f,t)$. This result will simplify the

remaining estimates.

Lemma III.1.7

$$\frac{\partial}{\partial t} W(\lambda,t,u) = \frac{\lambda}{p(t)} W(\lambda,t,u)(u-t) \quad , \tag{3.6}$$

where $p(t) = t(1+ct)$.

Proof.

Recall that $\phi_\lambda(x)$ can be expanded in Taylor series. Hence we have

$$\phi_\lambda(x) = \sum_{k=0}^{\infty} \frac{\phi_\lambda^{(k)}(0)}{k!} x^k \quad ,$$

and

$$\phi_{\lambda+c}(x) = \sum_{k=0}^{\infty} \frac{\phi_{\lambda+c}^{(k)}(0)}{k!} x^k \quad .$$

From the condition

$$-\phi_\lambda^{(k)}(x) = \lambda\phi_{\lambda+c}^{(k-1)}(x) \qquad \text{and} \qquad \phi_\lambda(0) = 1 \ ,$$

we have

$$\begin{aligned}
\phi_{\lambda+c}^{(k)}(0) &= (-1)^k(\lambda+c)(\lambda+2c) \ \ldots \ (\lambda+kc) \\
&= (-1)^k\lambda(\lambda+c) \ \ldots \ (\lambda+(k-1)c)\left(\frac{\lambda+kc}{\lambda}\right) \\
&= \phi_\lambda^{(k)}(0) \ \frac{\lambda+kc}{\lambda} \quad .
\end{aligned}$$

Hence,

$$\phi_{\lambda+c}(x) = \sum_{k=0}^{\infty} \frac{\phi_\lambda^{(k)}(0)}{k!} (1 + \frac{k}{\lambda} c)\ x^k$$

$$= \phi_\lambda(x) - cx \sum_{k=1}^{\infty} \frac{\phi_{\lambda+c}^{(k-1)}(0)}{(k-1)!} x^{k-1}$$

$$= \phi_\lambda(x) - cx\phi_{\lambda+c}(x) \quad .$$

or,

$$\phi_{\lambda+c}(x) = \frac{1}{1+cx} \phi_\lambda(x) \quad .$$

Now

$$\phi_{\lambda+c}^{(k)}(x) = (-1)^k(\lambda+c) \ \dots \ (\lambda+kc)\phi_{\lambda+c}(x)$$

$$= (-1)^k\lambda(\lambda+c) \ \dots \ (\lambda+(k-1)c)\phi_\lambda(x) \frac{1}{1+cx} (\frac{\lambda+kc}{\lambda})$$

$$= \phi_\lambda^{(k)}(x)(1 + \frac{k}{\lambda}) \frac{1}{1+cx} \ .$$

Thus,

$$\frac{\partial}{\partial t} W(\lambda,t,u) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \frac{k}{t} \phi_\lambda^{(k)}(t)\delta(u - \frac{k}{\lambda})$$

$$- \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \lambda\phi_{\lambda+c}^{(k)}(t)\delta(u - \frac{k}{\lambda})$$

$$= \frac{\lambda}{t} W(\lambda,t,u)u -$$

$$- \frac{\lambda}{1+ct} \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (1 + \frac{k}{\lambda} c)\phi_\lambda^{(k)}(t)\delta(u - \frac{k}{\lambda})$$

$$= \frac{\lambda}{t} W(\lambda,t,u)u - \frac{\lambda}{1+ct} W(\lambda,t,u) -$$

$$- \frac{\lambda c}{1+ct} W(\lambda,t,u)u$$

$$= \frac{\lambda}{t(1+ct)} W(\lambda,t,u)(u-t) .$$

(Q.E.D.)

Following Lemma III.1.7, one can easily deduce by induction a result similar to Lemma II.1.8. The first step of the induction is justified since $M_\lambda(x^i,t) = t^i$ for $i = 0,1$ (see Baskakov [3], p. 249).

Corollary III.1.8

If $A_m(\lambda,t) = \lambda^m \int_0^\infty W(\lambda,t,u)(u-t)^m du \quad m = 0,1,2,\ldots$,

then the following holds

(1) $$A_{m+1}(\lambda,t) = m\lambda p(t)A_{m-1}(\lambda,t)+p(t) \frac{d}{dt} A_m(\lambda,t) \quad ;$$

(2) $A_m(\lambda,t)$ is a polynomial in λ and t;

(3) the degree of $A_m(\lambda,t)$ in λ is $[\frac{m}{2}]$;

(4) the coefficient of λ^m in the polynomial $A_{2m}(\lambda,t)$ is $c_1\, p(t)^m$ and in the polynomial $A_{2m+1}(\lambda,t)$ is $c_2(1+2ct)p^m(t)$, where $p(t) = t(1+ct)$.

The next lemma provides the estimates needed in the proof of Lemma III.1.6.

Lemma III.1.9

If $g \in C_o[0,\beta)$, supp $g \subset (a,b) \subset (0,\beta)$, and $M > 0$ is

a fixed integer, then for $\gamma = 0,1,2,\ldots,M$, the following equation holds.

(3.7) $$\int_a^b \int_0^\beta W(\lambda,t,u)g(u)t^\gamma dt\, du =$$

$$= \sum_{\frac{m}{n}\in(a,b)} g(\tfrac{m}{\lambda})\frac{(m+\gamma)!}{m!} \frac{1}{(\lambda-c)(\lambda-2c)\ \ldots\ (\lambda-(\gamma+1)c)} .$$

Proof:

First, we prove the following assertion:

(3.8) $$(-1)^m \int_0^\beta \phi_\lambda^{(m)}(t)t^k dt = \frac{k!}{(\lambda-c)(\lambda-2c)\ \ldots\ (\lambda-(k-m+1)c)}$$

for non-negative integers k and m such that $k-m \leq M$.

The proof is by induction. First, from

$\phi_\lambda^{(k-1)}(t) = -\frac{1}{\lambda-c}\,\phi_{\lambda-c}^{(k)}(t)$, we have

$$\int_0^\beta \phi_\lambda(t)dt = \int_0^\beta -\frac{1}{\lambda-c}\,\phi'_{\lambda-c}(t)dt = \frac{1}{\lambda-c}\,\phi_{\lambda-c}(0) = \frac{1}{\lambda-c} .$$

Hence, (3.8) holds for $m = 0$, $k = 0$. For $m = 0$ fixed, we proceed by induction on k. Suppose $k+1 \leq M$, using conditions (4), (2) and (5) in Definition III.1.1 and the integration by parts, we have

$$\int_0^\beta \phi_\lambda(t)t^{k+1}dt = \int_0^\beta -\frac{1}{\lambda-c}\ \phi'_{\lambda-c}(t)\ t^{k+1}dt$$

$$= \frac{k+1}{\lambda-c} \int_0^\beta \phi_{\lambda-c}(t)t^k\, dt$$

$$= \frac{k+1}{\lambda-c}\,\frac{k!}{(\lambda-2c)\ \ldots\ (\lambda-(k+2)c)} = \frac{(k+1)!}{(\lambda-c)\ \ldots\ (\lambda-(k+2c)}$$

by the induction hypothesis. Therefore, (3.8) is valid for $m = 0$ and $k = 0,1,2,\ldots,M$.

Now, assume (3.8) holds for $m-1$ and all positive integers $k \leq M$. We first show by induction that (3.8) holds for m and $k-m \leq M-1$. By using a similar technique to above, we have

$$\int_0^\beta \phi_\lambda^{(m)}(t)\ t^k\ dt = -\int_0^\beta \lambda\phi_{\lambda+c}^{(m-1)}(t)\ t^k\ dt$$

$$= (-1)^m\ \lambda\ \frac{k!}{(\lambda-c)\ \ldots\ (\lambda-(k-m+1)c)}\ .$$

Next, for m and $k-m = M$, by using conditions (4), (2) and (5) of Definition III.1.1, the integration by part, and the induction hypothesis, we obtain

$$\int_0^\beta \phi_\lambda^{(m)}(t)\, t^{m+M}\ dt = -\int_0^\beta \lambda\ \phi_{\lambda-c}^{(m-1)}(t)\ t^{m+M}\ dt$$

$$= \lambda(m+M)\int^\beta \phi_{\lambda+c}^{(m)}\ t^{m+M-1}\ dt$$

$$= (-1)^m\ \lambda(m+M)\ \frac{(m+M-1)!}{\lambda(\lambda-c)\ \ldots\ (\lambda(M-1)c)}\ .$$

Hence the proof of equation (3.8) is completed.

Using relation (3.8) for $k = m+\gamma$, $\gamma = 0,1,2,\ldots,M$, we have

$$\int_0^\beta W(\lambda,t,u)t^\gamma\ dt = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\int_0^\beta \phi_\lambda^{(m)}(t)t^{m+\gamma}\delta(u-\frac{m}{\lambda})\ dt$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\ \frac{(m+\gamma)!}{(\lambda-c)(\lambda-2c)\ \ldots\ (\lambda-(\gamma+1)c)}\delta(u-\frac{m}{\lambda})\ .$$

Hence,

$$\int_a^b \int_0^\infty g(u)\ W(\lambda,t,u)t^\gamma\ dt\ du$$

$$= \sum_{\frac{m}{\lambda}\in(a,b)} g(\tfrac{m}{\lambda})\ \frac{1}{m!}\ \frac{(m+\gamma)!}{(\lambda-c)(\lambda-2c)\ \ldots\ (\lambda-(\gamma+1)c)} \quad .$$

(Q.E.D.)

§1.3 Proofs of Lemmas III.1.3, III.1.4 and III.1.5

We first prove Lemma III.1.3

Let $h \in C^\infty(\Omega)$ be a function with compact support, such that $h(t) = 1$ on $[a-\delta,b+\delta]$ for some $\delta > 0$, and $\sup_{t\in\Omega} |h(t)| = 1$. Now assume $|f(t)| \leq M\ g(t)$.

If condition (i) in the lemma holds for functions with compact supports, then, for $t \in [a,b]$, we have

$$(3.9) \qquad S^*_\lambda(f,t)-f(t) = \int_\Omega W^*(\lambda,t,u)f(u)h(u)du - f(t)h(t) + \\ + \int_{\Omega-[a-\delta,b+\delta]} W^*(\lambda,t,u)[f(u)-f(u)h(u)]du \quad ,$$

and since

$$(3.10) \qquad \int_{\Omega-[a-\delta,b+\delta]} W^*(\lambda,t,u)[f(u)-f(u)h(u)]du \\ \leq 2M\ \delta^{-m} \int_\Omega W^*(\lambda,t,u)|u-t|^m\ g(u)du \equiv I(\delta)$$

from which using the Cauchy-Schwarz inequality we obtain

$$(3.11)\qquad I(\delta) \le 2M\delta^{-m}\Big[\int_\Omega W^*(\lambda,t,u)(u-t)^{2m}du\Big]^{1/2}\Big[\int_\Omega W^*(\lambda,t,u)g^2(u)du\Big]^{1/2}$$

$$= O(\phi_\lambda^{(1/2)}(t)) \quad .$$

Hence (i) is also true for functions $|f(t)| \le Mg(t)$. (For the case of uniform convergence, it is enough to notice that the estimate in (3.11) is uniform in $[a,b]$.

Now suppose $\|S^*_{\lambda_n}(f,t)-f(t)\|_{C[a,b]} = O(\sigma(\lambda_n))$ <or $o(\sigma(\lambda_n))$>, $|f(t)| \le M\, g(t)$, and assume (ii) is true for functions with compact supports. Choosing h as above, we have for $t \in [a,b]$

$$(3.12)\qquad S^*_\lambda(fh,t)-f(t)h(t) = S^*_\lambda(f,t)-f(t) +$$

$$+ \int_{\Omega-[a-\delta,b+\delta]} W^*(\lambda,t,u)[f(u)h(u)-f(u)]du \quad .$$

Since

$$(3.13)\qquad \Big\| \int_{\Omega-[a-\delta,a+\delta]} W^*[f(u)h(u)-f(u)]du\Big\|_{C[a,b]}$$

$$\le M'\|\phi_\lambda(t)\|^{1/2}$$

$$= o(\sigma(\lambda)) \quad ,$$

we obtain that, if $\|S^*_{\lambda_n}(f,t)-f(t)\|_{C[a,b]} = O(\sigma(\lambda_n))$ <resp. $o(\sigma(\lambda_n))$> , then

$$(3.14)\qquad \|S^*_{\lambda_n}(fh,t)-f(t)g(t)\|_{C[a,b]} = O(\sigma(\lambda_n)) \quad \langle \text{resp. } o(\sigma(\lambda_n))\rangle .$$

Therefore it follows that

$$fh|_{[a,b]} = f|_{[a,b]} \in L_{[a,b]}$$

by the hypothesis.

Part (iii) can be proved similarly.

Lemma III.1.4 follows readily from the Corollary III.1.8. We note that, since $\lambda^{-m} A_m(\lambda,t)$, and hence $M_n(x^m,t)$ are polynomials in t and λ^{-1}, $M_n(\psi_A(x),t)$ is uniformly bounded, where $\psi_A(x) = (1+x)^A$. Therefore, the dominated function in the theorem can be chosen at least as $\psi_A(x)$ for any $A > 0$.

The proof of Lemma III.1.5 follows the same arguments as Lemma II.1.4. Let the numbers $\alpha(j,k)$ be defined by

$$M_{2\lambda}(f,k,t)-M_\lambda(f,k,t) = \sum_{j=0}^{k+1} \alpha(j,k) M_{2^j\lambda}(f,t) \quad .$$

Clearly, these are the same numbers as defined in equation (2.10) and so Lemma II.1.9 can be applied. Using Lemma II.1.9 and Corollary III.1.7, the proof of Lemma II.1.4 contained in Chapter II, Section 1.4 can be appropriately modified to obtain Lemma III.1.5.

§1.4 Proof of Lemma III.1.6

The proof of Lemma III.1.6 will be similar to the proof of Lemma II.2.5. As in step 1° of the proof of Lemma II.2.5, we can write

$$\lambda^{k+1} < [M_{2\lambda}(f,k,\cdot)-M_{\lambda}(f,k,\cdot)],g(\cdot)>$$

$$= \sum_{\gamma=0}^{2k+3} I_{\gamma} + o(1)||f||_{(A)}$$

where

$$(3.15)\quad \begin{cases} I_{\gamma} = \lambda^{k+1} \int_a^b \int_0^{\infty} \sum_{j=0}^{k+1} \alpha(j,k)W(2^j\lambda,t,u)\psi_{\gamma}(u)t^{\gamma}dtdu \\ \psi_{\gamma}(u) = \sum_{m=\gamma}^{2k+2} (-1)^{m-\gamma} \frac{1}{m!} f(u)g^{(m)}(u)u^{m-\gamma} \end{cases}$$

$$\gamma = 0,1,\ldots,2k+2$$

$$(3.16)\quad \begin{cases} I_{2k+3} = \lambda^{k+1} \int_a^b \int_0^{\infty} \sum_{j=0}^{k+1} \alpha(j,k)W(2^j\lambda,t,u)f(u)\varepsilon(t,u)(t-u)^{2k+2}dtdu \\ |\varepsilon(t,u)| \leq \frac{2}{(2k+2)!} ||g^{(2k+2)}||_{C[a,b]} \quad . \end{cases}$$

Since

$$|I_{2k+3}| \leq M_g||f||_{(A)} \lambda^{k+1} \int_a^b \int_0^{\infty} \sum_{j=0}^{k+1} |\alpha(j,k)|W(2^j\lambda,t,u)(t-u)^{2k+1}dtdu \ ,$$

in order to show $I_{2k+3} = 0(1)$, it is sufficient to show

$$(3.17)\quad F_{\lambda}(k) = \lambda^k \int_a^b \int_0^{\infty} W(\lambda,t,u)(u-t)^{2k}dtdu = 0(1) \quad .$$

First, $F_{\lambda}(k)$ can be rewritten as

$$F_\lambda(k) = \int^{b+1} \int_a^b W(\lambda,t,n)(u-t)^{2k}\, du\, dt +$$

$$+ \lambda^k \int_{b+1}^{\infty} \int_a^b W(\lambda,t,u)(u-t)^{2k}\, du\, dt$$

$$= J_1 + J_2 \quad .$$

The fact that $J_1 = 0(1)$ follows from Corollary III.1.7, as the integration in t is only over a finite interval.

To estimate J_2, we notice that when $t \geq b+1$, $0 < t-b \leq t-u$. Hence,

$$(3.18) \qquad J_2 \leq \lambda^k \int_{b+1}^{\infty} \int_{a^+}^{b^-} W(\lambda,t,u) \frac{(u-t)^{2(k+N)}}{(t-b)^{2N}}\, du\, dt$$

$$\leq \lambda^k \int_{b+1}^{\infty} \frac{dt}{(t-b)^{2N}} \left\{\int_{a^+}^{b^-} W(\lambda,t,u)(u-t)^{4k+2} du\right\}^{1/2} \cdot$$

$$\cdot \left\{\int_{a^+}^{b^-} W(\lambda,t,u)(u-t)^{4N-2} du\right\}^{1/2} \quad .$$

Recall that $p(t) = t(1+ct)$ is a polynomial of degree less than or equal to 2, and therefore $\lambda^{-m} A_m(\lambda,t) = \int_0^\infty W(\lambda,t,u)(u-t)^m\, dt$ is a polynomial in t with degree less than or equal to m. Moreover, $\lambda^{-m} A_m(\lambda,t) = 0(\lambda^{-[m+1/2]})$. Therefore,

$$[\int_a^b W(\lambda,t,u)(u-t)^{4N-2} du]^{1/2} \leq M\lambda^{-(N-\frac{1}{2})} (P_{4N-2}(t))^{1/2} \quad ,$$

where $P_{4N-2}(t)$ is a polynomial in t of degree $4N-2$. Hence (3.18)

follows

$$J_2 \leq \lambda^{K-N+\frac{1}{2}} \int_{b+1}^{\infty} \frac{(P_{4N-1}(t))^{1/2}}{(t-b)^{2N}} \left\{ \int_a^b W(\lambda,t,u)(u-t)^{4k+2} du \right\}^{\frac{1}{2}} dt \ .$$

By Cauchy-Schwarz inequality, we can further estimate

$$J_2 \leq \lambda^{k+\frac{1}{2}-N} \left\{ \int_{b+1}^{\infty} \frac{P_{4N-2}(t)}{(t-b)^{4N}} dt \right\}^{\frac{1}{2}} \left\{ \int_{b+1}^{\infty} \int_{a^+}^{b^-} W(\lambda,t,u)(u-t)^{4k+2} du dt \right\}^{\frac{1}{2}}$$

$$\equiv \lambda^{k+\frac{1}{2}-N} L_1 \cdot L_2 \ .$$

It is sufficient to show L_1^2 and L_2^2 are finite integrals, for then by choosing $N \geq k+1$, $J_2 = O(1)$.

The estimate $L_1^2 < \infty$ is trivial, since the integer and $\frac{P_{4N+2}(t)}{(t-b)^{4N}}$ is dominated by Mt^{-2} for some $M > 0$, and $\int_{b+1}^{\infty} t^{-2} dt$ is convergent. To estimate L_2^2, following (3.7), we have

$$F \equiv \int_{(a,b)} \int_0^{\infty} u^{4k+2-j} W(\lambda,t,u) t^j dt\, du \qquad (o \leq j \leq 4k+2)$$

$$= \frac{1}{n} \sum_{\frac{m}{\lambda} \in (a,b)} \left(\frac{m}{\lambda}\right)^{4k+2-j} \frac{(m+j)!}{\lambda^j m!} \frac{\lambda^{j+1}}{(\lambda-c) \ \ldots \ (\lambda(j+1)c)} \ .$$

Since

$$\left| \frac{\lambda^{j+1}}{(\lambda-x) \ \ldots \ (\lambda-(j+1)c)} \right| \leq 2$$

and, for $m \sim \lambda$,

$$\frac{(m+j)\ \ldots\ (m+1)}{\lambda^j} \le \left(\frac{m+j}{\lambda}\right)^j \le 2^{4k+2}\left(\frac{m}{\lambda}\right)^j ,$$

it follows that

$$F \le 2^{4k+3}\frac{1}{\lambda}\sum_{\frac{m}{\lambda}\in(a,b)}\left(\frac{m}{\lambda}\right)^{4k+2}$$

$$= 2^{4k+3}\left[\int_a^b u^{4k+2}\,du + R\right]$$

where, by estimating integrals by the rectangular formula

$$|R| \le \frac{(b-a)^2}{2\lambda}\left|\left|\frac{d}{du}u^{4k+2}\right|\right|_{C[a,b]} .$$

Hence, $F = O(1)$. Therefore, L_2 and J_2 are $O(1)$. In other words, $I_{2k+3} = O(1)$.

The estimate of I_γ, $\gamma \le 2k+2$, is similar to the estimate for the case of Bernstein Polynomials. We use the Euler-McLaurin formula. By equation (3.7),

$$I_\gamma = \lambda^{k+1}\sum_{j=0}^{k+1}\alpha(j,k)\sum_{\frac{m}{\lambda_j}\in(a,b)}\psi_r\left(\frac{m}{\lambda_j}\right)\frac{(m+\gamma)!}{m!}\sum_{\ell=1}^{\gamma+1}(\lambda_j-\ell c)^{-1}$$

$$(\lambda_j = 2^j\lambda) .$$

Further, we can write

$$\frac{(m+\gamma)!}{\lambda_j^{\gamma}\, m!} = \prod_{i=1}^{\gamma} \left(\frac{m}{\lambda_j} + \frac{i}{\lambda_j}\right) = \sum_{v=0}^{k} c_v \left(\frac{m}{\lambda_j}\right)^{\gamma-v} \lambda_j^{-v} + O(\lambda_j^{-k-1})$$

$$\lambda_j^{\gamma+1} \prod_{\ell=1}^{\gamma+1} (\lambda_j - \ell c)^{-1} = \prod_{\ell=1}^{\gamma+1} \left(1 - \frac{\ell c}{\lambda_j}\right)^{-1} = \sum_{v=0}^{k} d_v \lambda_j^{v} + O(\lambda_j^{-k-1})$$

where c_v and d_v are absolute constants. In other words, we can express I as

$$I_\gamma = \lambda^{k+1} \sum_{j=0}^{k+1} \alpha(j,k) \sum_{\frac{m}{\lambda_j} \in (a,b)} \psi_\gamma\left(\frac{m}{\lambda_j}\right) \frac{1}{\lambda_j} \left\{ \sum_{v=0}^{k} q_v\left(\frac{m}{\lambda_j}\right) \lambda_j^{v} + O(\lambda_j^{-k-1}) \right\}$$

where q_v are polynomials.

Using the Euler-McLaurin formula as in the corresponding estimate for the case of Bernstein Polynomials we obtain

$$I_\gamma = O(1) \quad .$$

(Q.E.D)

Remarks III.1.10

1° The saturation theorem for Baskakov operators which we have just proved is applicable to functions with growth not faster than some $\psi_A(t) = (1+t)^A$. The saturation theorem for the Szasz operator proved in Chapter II is applicable for functions with growth not faster than e^{At}. Hence, the theorem proved in this chapter does not contain the result for Szasz operators proved in the last chapter.

2° The conditions in Definitions III.1.1 are slightly different from those in the original definition (c.f., [2] and [54]). The differences are, the corresponding intervals in conditions (1), (3) and (4) have been changed to $[0,\beta)$, where β satisfies condition (5).

These modifications are based on the concrete examples. For instance, in case of Bernstein polynomials, $M_n(f,x) = B_n(f,x)$, $\phi_n(x) = (1-x)^n$, $\beta = 1$. In this case, $\phi_n(x)$ would not satisfy condition (3): $(-1)^k \phi_n^{(k)}(x) \geq 0$, if $x > 1$.

3° The condition (5) in Definition III.1.1 is in fact equivalent to one of the conditions, i.e.,

$$(3.19) \qquad \int_0^{E(c)} (-1)^\ell \frac{\phi_\lambda^{(\ell)}(x)}{\ell!} x^\ell \, dx = \frac{1}{\lambda-c}, \quad \ell = o,1,2,\ldots$$

used by Suzuki in his saturation theorem for Baskakov operators ([54], p. 441). The condition (3.13), as being equivalent to condition (5) of Definition III.1.1, is enough for proving the saturation theorem, while the other conditions, i.e., conditions (20), part of (21), and (22) of [54], are redundant for this purpose.

We would like to point out that the notation $E(c)$ used by Suzuki is equal to the "β" in our definition. The value of $E(c)$ is equal to ∞ when $c \geq 0$ and is equal to $\frac{1}{|c|}$ when $c < 0$. The formula that "$E(c) = \frac{5c^2-c-2}{2c(c-1)}$ if $c \neq 0,1$" given by Suzuki is valid only if $c = -1$.

We shall show in the next proposition the equivalence of conditions (5) in Definition III.1.1 and (3.19).

Proposition III.1.11

If $\{\phi_\lambda(x)\}$ is a family of functions defined in Definition III.1.1, then the following are equivalent:

(1) for any fixed $M > 0$,

$$\lim_{t \to \beta} \phi_\lambda(t) t^k = 0, \quad k = 0,1,\ldots,M \;;$$

(2) $$(-1)^m \int_0^\beta \frac{\phi_\lambda^{(m)}(t)}{m!} t^m \, dt = \frac{1}{\lambda - c}, \quad m = 0,1,2,\ldots \;;$$

(3) for any fixed M,

$$(-1)^m \int^\beta \phi_\lambda^{(m)}(t) \; t^{m+r} \, dt$$

$$= \frac{(m+r)!}{(\lambda-c)\ \ldots\ (\lambda-(r+1)c)}, \quad r = 0,1,2,\ldots,M, \quad m = 0,1,2,\ldots \;.$$

Proof.

The implication (3) ==> (2) is trivial, and the implication (1) ==> (3) has been shown in equation (3.8). It remains to show the implication (2) ==> (1). First, for $k = 0$, we must show $\lim_{t \to \beta} \phi_\lambda(t) = 0$. This fact follows from conditions (4) and (2) of Definition III.1.1 and the following simple calcualtion:

$$\frac{1}{\lambda} = \int_0^\beta \phi_{\lambda+c}(t) \, dt = -\frac{1}{\lambda} \int_0^\beta \phi_\lambda'(t) \, dt$$

$$= -\frac{1}{\lambda} \phi_\lambda(t) \Big|_0^\beta = \frac{1}{\lambda} - \frac{1}{\lambda} \lim_{t \to \beta} \phi_\lambda(t) \quad ,$$

and $\lim_{t \to \beta} \phi_\lambda(t)$ readily follows.

Next, for $0 < k \leq M$, where M is an arbitrary but fixed constant, using integration by parts, we obtain

$$\text{(3.20)} \qquad \frac{1}{\lambda-kc} = (-1)^k \int_0^\beta \frac{\phi^{(k)}_{\lambda-(k-1)c}(t)\ t^k}{k!}\, dt$$

$$= \frac{(-1)^k \phi^{(k-1)}_{\lambda-(k-1)c}(t)\ t^k}{k!} \Bigg|_0^\beta$$

$$+ \frac{(-1)^{k-1}}{(k-1)!} \int_0^\beta \phi^{(k-1)}_{\lambda-(k-1)c}(t)\ t^{k-1}\, dt \quad .$$

From the condition $(-1)\phi^{(m)}_\lambda(t) = \phi^{(m-1)}_{\lambda+c}(t)$, we have $(-1)^{k+1}\phi^{(k-1)}_{\lambda-(k-1)c}(t) = \lambda(\lambda-c)\ \ldots\ (\lambda-(k-1)c)\phi_\lambda(t)$. Also

$$\frac{(-1)^{k+1}}{(k-1)!} \int_0^\beta \phi^{(k-1)}_{\lambda-(k-1)c}(t)\ t^{k-1}\, dt = \frac{1}{\lambda-kc} \quad .$$

Using these facts and the condition $\phi_\lambda(0) = 1$, we obtain from equation (3.20) that $\lim\limits_{t \to \beta} \phi_\lambda(t)\ t^k = 0$.

(Q.E.D.)

4° We do not intend to say that condition (5) of Definition III.1.1 is of more natural form than the condition (3.19) ($E(c) \equiv \beta$). The additional condition (5) here was introduced in the study of Baskakov's operator directly from the original work of Baskakov, and is independent of Suzuki's work. However, we would like to note that Suzuki is the first one who noticed the necessity to introduce an extra condition.

§2. A General Linear Combination

In his paper [8], Butzer asked whether there exists other linear combinations of degree not exceeding $2^k n$ approaching $f(x)$ more closely than the combination $B_n(f,k,t)$.

We shall give an anser to this problem in the present section.

For convenience, we refer to the operators discussed in Chapter I together with the Baskakov operators as the <u>Bernstein-type operators</u>.

<u>Definition III.2.1</u>

Let $S_\lambda(f,t)$ be any of the Bernstein-type operators. Let $d_0, d_1, \ldots, d_k$ be $(k+1)$ distinct positive integers, and define

$$C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} , \tag{3.21}$$

and

$$S_\lambda(f,k,t) = \sum_{j=0}^{k} C(j,k)\, S_{d_j\lambda}(f,t) . \tag{3.22}$$

We state the following condition:

<u>Definition III.2.2</u>

A function f is said to satisfy condition A if $f \in C(\Omega)$ (for Bernstein polynomials, $\Omega = [0,1]$, for Baskakov operators. $\Omega = [0,\beta)$, and for the other operators, $\Omega = [0,\infty)$), $|f(t)| \leq M\psi_A(t)$ (for Baskakov operators, $\psi_A(t) = (1+t)^A$, for

other operators, $\psi_A(t) = e^{At}$), for some $A > 0$.

We have the following

Proposition III.2.3

Suppose f satisfies condition A and let $[a,b] \subset \Omega$. Then

1. If $f^{(2k+2)}(\xi)$ exists, then

$$|S_\lambda(f,k,\xi)-f(\xi)| = O(\lambda^{-(k+1)}) \ .$$

2. If $f \in C^{2k+2}[a,b]$, then the above relation is uniform in any subinterval $[a_1,b_1] \subset (a,b)$.

3. If $f^{(2k+2)}(\xi)$ exists, then

$$\lambda^{k+1}[S_\lambda(f,k,t)-f(t)]$$

$$= \sum_j Q(j,k,t) f^{(j)}(t) + o(1)$$

where $Q(j,k,t)$ are polynomials in t such that $Q(2k+2,k,t) = c_1\ p(t)^{k+1}$ and $Q(2k+1,k,t) = c_2\ p(t)^k\ p'(t)$.

This proposition would become clear by the following lemma.

Lemma III.2.4

$$(3.23)\qquad \sum_{j=0}^{k} C(j,k) d_j^{-m} = \begin{cases} 1 & m = 0 \\ 0 & m = 1,2,\dots,k \ . \end{cases}$$

Proof

Consider the Lagrange Polynomial

$$L_{k,m}(x) = \sum_{i=0}^{k} x_i^m \ell_i(x) - x^m$$

where

$$\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{k} \frac{x_j - x}{x_j - x_i} , \qquad x_i = d_i^+ \quad .$$

Since $L_{k,m}(x_i) = 0$ for $i = 0,1,\ldots,k$, and $L_{k,m}(x_i)$ is a polynomial of degree k, $L_{k,m}(x) \equiv 0$. In particular

$$\sum_{i=0}^{k} C(i,k) \frac{1}{d_i^m} = \begin{cases} L_{k,m}(0) = 0 & m \neq 0 , \\ L_{k,0}(0)+1 = 1 & m = 0. \end{cases}$$

(Q.E.D.)

The proof of Proposition III.2.3 is similar to the proof of Lemma II.1.4: expand $f(u)$ to the Taylor polynomial at point t, and then use the fact that $\lambda^{-m} A_m(\lambda,t) \equiv \int_\Omega W(\lambda,t,u)(-u-t)^{2m}\,du$ is a polynomial in λ^{-1}, and Lemma III.2.4. The details will be omitted.

Remarks III.2.5

1° If we choose $d_j = 2^j$, the combinations defined in (3.23) reduce to the combinations investigated by Butzer in [8], (c.f., (2.1)). If, however, we choose $d_j = j+1$, $j = 0,1,\ldots,k$ and use Bernstein polynomials $B_n(f,k)$ in place of $S_\lambda(f,t)$, we obtain a polynomial $S_n(f,k,t)$ which is a linear combination of $B_n(f,t)$, of degree $(k+1)n$, but approaching $f(t)$ with the same rate as

$B_n(f,k,t)$ (defined in (2.1), which is of degree $2^k n$). Hence, the Proposition III.2.3 gives a positive answer to the problem raised in [8].

2° A similar saturation result is also true for this "general" combinations, but we cannot prove it here.

To state the difficulties, we need a slightly more general notation. Let us denote by $S_\lambda(f,t;d_0,d_1,\ldots,d_k)$ the combination defined in (3.23) corresponding to $d_0,d_1,\ldots,d_k$. Then, although we do have a recursion formula

$$S_\lambda(f,t;d_0,\ldots,d_{k+1}) = \frac{1}{d_{k+1}-d_0}\{d_{k+1}\, S_\lambda(f,t;d_1,\ldots,d_{k+1}) -$$

$$- d_0 S_\lambda(f,t;d_0,\ldots,d_k)\} \quad ,$$

we are unable to prove (directly) the induction step. That is, from

$$||S_\lambda(f,t;d_0,\ldots,d_{k+1})-f(t)|| = O(\lambda^{-(k+1)}) \quad ,$$

we cannot deduce (directly)

$$\|S_\lambda(f,t,d_{i_0},\ldots,d_{i_k})-f(t)\| = O(\lambda^{-k})$$

for any $d_{i_0},\ldots,d_{i_k}$. Thus, we cannot conclude the existence of $f^{(2k)}$ as out intermediate step.

However, this problem can be solved by using the result of the next chapter. The procedure may look odd from a logical point of view, namely, before proving the saturation problem (the optimal case), we prove the inverse problem (the non-optimal case); but in

proving the non-optimal case, we need one direction of the result in the optimal case (the easy direction), the result stated in Proposition III.2.3. After proving the non-optimal result, we are able to conclude $f \in C^{2k}$ from

$$||S_\lambda(f,t;d_0,\dots,d_k)-f(t)|| = O(\lambda^{-(k+1)}) \quad ,$$

and this exactly is what we need for proving the optimal case.

CHAPTER IV

THE INVERSE THEOREM

§1 Definitions and the Inverse Result

Let $S_\lambda(f,t)$ be the Bernstein-type operator. Let $S_\lambda(f,k,t)$ be a linear combination corresponding to $d_0,d_1,\ldots,d_k$ defined by Definition III.2.1. That is,

$$S_\lambda(f,k,t) = \sum_{j=0}^{k} c(j,k)S_{d_j\lambda}(f,t) \quad , \tag{4.1}$$

where

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} \quad . \tag{4.2}$$

We have seen [Proposition III.2.3] that $S_\lambda(f,k,t)$ converges to $f(t)$ with the optimal degree $\lambda^{-(k+1)}$. This order of convergence can actually be used, at least for some special combinations, to determine the smoothness of a function.

In this section, we shall discuss a way of measuring the smoothness of functions when the exact degree of convergence is slower than the critical order of convergence for $S_\lambda(f,k,t)$. This is the inverse problem.

<u>Definition IV.1.1</u>

Suppose the optimal rate of convergence for a family of

operators $S_\lambda(f,t)$ converging to $f(t)$ is $q(\lambda)$. The non-optimal problem, or the inverse problem, is to determine the class of functions f such that $S_\lambda(f,t)$ converges to $f(t)$ with the rate $q(\lambda)^\beta$ for each β, $0 < \beta < 1$.

Inverse problems are, in general, more difficult to solve than the direct problem. In the direction of solving the inverse problems for Bernstein-type operators, Berens and Lorentz [5] have already proved an inverse theorem for Bernstein Polynomials (not for combinations). Their result is:

Theorem IV.1.2

Let $0 < \alpha < 2$, $f \in C[0,1]$, and $B_n(f,t)$ be the Bernstein polynomials. Then $\|p(t)^{-1}[B_n(f,t)-f(t)]\|_{C[0,1]} = O(n^{-\alpha/2})$ if and only if $f \in \mathrm{Lip}^*(\alpha;0,1)$, where $p(t) = t(1-t)$, and $\mathrm{Lip}^*(\alpha;0,1) = \{f \in C[0,1];\ \sup_{|t|\leq h} |f(\cdot+t)-2f(\cdot)+f(\cdot-t)| \leq Mh^\alpha\}$ is the Zygmund class.

Since the above theorem is a global result, it would be interesting to see a local inverse theorem; that is, to determine the local smoothness of functions from the local convergence properties of $B_n(f,t)$. Another question is, can we obtain a similar result for "general" combinations of Bernstein-type operators?

We would like to have complete answers for the above questions. However, the solutions we have obtained so far do not fit the Phillips operator, which is defined as

$$S_\lambda(f,t) = \int_0^\infty e^{-\lambda(t+u)} \sum_{n=1}^\infty \frac{(\lambda^2 t)^n u^{n-1}}{n!\,(n-1)!} f(u)\,du + e^{-\lambda t} f(0) \quad .$$

For later reference, we use $S_\lambda^i(f,t)$ to refer to

$$
(4.3)\quad \begin{cases}
S_\lambda^1(f,t) = e^{-\lambda t}\displaystyle\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}f(\tfrac{k}{\lambda}), & \text{the Szasz operators;}\\[2ex]
S_n^2(f,t) = \dfrac{1}{(n-1)!}(\tfrac{n}{t})^n\displaystyle\int_0^\infty e^{-nu/t}u^{n-1}f(u)du & ;\\[2ex]
S_n^3(f,t) = B_n(f,t), & \text{the Bernstein polynomials;}\\[2ex]
S^4(f,t) = \sqrt{\dfrac{\lambda}{2\pi}}\displaystyle\int_0^\infty e^{-\lambda(u-t)^2/2}f(u)\,du & ;\\[2ex]
S_\lambda^5(f,t) = \displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{\phi_\lambda^{(k)}(t)}{k!}t^k f(\tfrac{k}{\lambda}), & \text{the Baskakov}
\end{cases}
$$

operator defined in Definition III.1.1.

The notations in (4.3), except $S_n^3(f,t)$, are the same as in (2.19). That is, the notation for the Phillips operator in (2.19) is used for Bernstein polynomials here. However, this change will not cause any ambiguity, for we shall not deal with the Phillips operators in this chapter.

Before stating our result, we need the following notations and definitions.

<u>Definition IV.1.3</u>

1° A function f is said to satisfy condition A if $f \in C(\Omega)$ (for Bernstein polynomials, $\Omega = [0,1]$; for Baskakov operators, $\Omega = [0,\beta)$, and for the other operators, $\Omega = [0,\infty)$), $|f(t)| \le M\psi_A(t)$ (for Baskakov operators, $\psi_A(t) = (1+t)^A$, for

other operators, $\psi_A(t) = e^{At}$), for some $A > 0$.

2° The kernels of $S^i_\lambda(f,t)$ will be denoted by $W_i(\lambda,t,u)$. That is, $S^i_\lambda(f,t) = \int_\Omega W_i(\lambda,t,u)f(u)du$.

3° The various moduli of continuity, $\omega_k(f,h,a,b)$, are defined by induction as follows:

a) $$\Delta^1_h f(x) \equiv \Delta_h f(x) = f(x+h)-f(x)$$

$$\Delta^{k+1}_h f(x) = \Delta_h \Delta^k_h f(x) = \sum_{\gamma=0}^{k+1} (-1)^{k+1-\gamma} \binom{k+1}{\gamma} f(x+\gamma h)$$

b) $$\bar{\Delta}^1_h f(x) \equiv \bar{\Delta}_h f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$$

$$\bar{\Delta}^{k+1}_h f(x) = \bar{\Delta}_h \; \bar{\Delta}^k_h f(x)$$

and

$$\omega_k(f,h,a,b) = \sup\{|\Delta^k_t f(x)|;|t| \leq h, x, x+kt \in [a,b]\} . \tag{4.4}$$

4° For $0 < \alpha < 2$, the generalized Zygmund class is defined as

$$\text{Liz}\ (\alpha,k;a,b) = \{f;\ \omega_{2k}(f,h,a,b) \leq Mh^{\alpha k}\} . \tag{4.5}$$

From the above definitions, we observe two additional facts:

5° In equation (4.4), Δ can be replaced by $\bar{\Delta}$, that is, we also have

$$\omega_k(f,h,a,b) = \sup\ \{|\bar{\Delta}^k_t f(t)|;|t| \leq h, x \pm \frac{k}{2}\ t \in [a,b]\} . \tag{4.6}$$

Also,

$$\bar{\Delta}_h^{2m} f(x) = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} f(x+(m-i)h) \quad .$$

6° In (4.5), when $k = 1$, the class Liz $(\alpha,1)$ reduces to the usual Zygmund class $\mathrm{Lip}^* \alpha$.

We are now in a position to state our inverse theorem.

Theorem IV.1.4

Let $0 < a_i < a_{i+1} < b_{i+1} < b_i \in \Omega$, $i = 1,2$, $0 < \alpha < 2$, and suppose that f satisfies Condition A. Then in the following, the implications (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) hold.

(1) $\|S^i_\lambda(f,k,t)-f(t)\|_{C[a_1,b_1]} = O(\lambda^{-\frac{\alpha}{2}(k+1)})$;

(2) $f \in \mathrm{Liz}\ (\alpha,k+1;a_2,b_2)$;

(3) If $\tau = \alpha(k+1)$, then

(a) $m < \tau < m+1$, $m = 0,1,2,\ldots,2k+1$, implies $f^{(m)}$ exists and $f^{(m)} \in \mathrm{Lip}\ (\tau-m,a_2,b_2)$,

and

(b) $\tau = m+1$, $m = 0,1,2,\ldots,2k$, implies $f^{(m)}$ exists and $f^{(m)} \in \mathrm{Lip}^*\ (1,a_2,b_2)$;

(4) $\|S^i_\lambda(f,k,t)-f(t)\|_{C[a_3,b_3]} = O(\lambda^{-(\alpha/2)(k+1)})$.

The equivalence of (2) and (3) is known, (c.f., ([58], pp. 257, 333 and 337). The implication of (3) to (4), when τ is

not an integer, follows easily by applying Holder's inequality, but we have not found a similar way to show the equivalence when τ is an integer. We shall therefore prove the implication of (2) to (4) instead of proving it for the individual values of τ in (3).

In the following we shall use $S_\lambda(f,k,t)$ instead of $S^i_\lambda(f,k,t)$ if no ambiguity occurs.

The proof will be divided into two parts. We first prove the special case when f is of compact support strictly contained inside the open interval (a,b), and then pass to the general case.

The method used in proving the case when f has compact support is similar to the one used in the proof of [5] by Berens and Lorentz: We shall show, under the restriction that $\operatorname{supp} f \subset (a,b)$, that the conditions (1) and (2) in the theorem, with (a_i,b_i) being replaced by (a,b), are both equivalent to the fact that f belongs to an intermediate space $C_o(\alpha,k;a,b)$, which will be defined in the next section.

§2 The Space $C_o(\alpha,k;a',b')$

Let $[a,b]$ be a fixed interval, and let $[a',b'] \subset (a,b)$. Let G denote the class

(4.7) $$G = \{g;\ g \in C_o^{2k+2},\ \operatorname{supp} g \subset [a',b']\}$$

(where C_o^{2k+2} is the space of $2k+2$-continuously differentiable functions with compact supports).

Further, for functions $f \in C_o$ (space of continuous functions with compact supports) with $\operatorname{supp} f \subset [a',b']$, we define a Peetre K-function by

$$(4.8) \qquad K(\xi,t) = \inf_{g \in G} \{ ||f-g|| + \xi(||g||+||g^{(2k+2)}||) \}$$

where $0 < \xi \leq 1$. The norms in (4.8) are the supremum-norms on $[a',b']$.

A continuous function f with compact support in $[a',b']$ is said to belong to $C_o(\alpha,k+1;a',b')$, $0 < \alpha < 2$, if

$$(4.9) \qquad ||f||_\alpha \equiv \sup_{0<\xi\leq 1} \xi^{-\alpha/2} K(\xi,f)$$

is finite.

We shall show in this section that, under the restriction that f has compact support strictly in (a,b), the condition (1) in Theorem IV.1.4 (with a_1 and b_1 being replaced by a and b respectively) is equivalent to the fact that f belongs to $C_o(\alpha,k+1;a',b')$ for some $[a',b'] \subset (a,b)$.

First observe that, from the condition $\operatorname{supp} f \subset (a,b)$, we can choose a', a'', b' and b'' in such a way that $a < a' < a'' < b'' < b' < b$ and $\operatorname{supp} f \subset [a'',b'']$.

We begin with some estimates involving the K-function defined in (4.8).

<u>Lemma IV.2.1</u>

Let $a < a' < a'' < b'' < b' < b$. If $f \in C_o$ with

supp f $\subset [a'',b'']$, and $\|S_\lambda(f,k,t)-f(t)\|_{C[a,b]} \leq M\,\lambda^{-\frac{\alpha}{2}(k+1)}$,

then

$$K(\xi,t) \leq M_0[\lambda^{-\frac{\alpha}{2}(k+1)} + \lambda^{k+1}\,\xi K(\lambda^{-(k+1)},f)] \quad , \tag{4.10}$$

Proof:

Since supp f $\subset [a'',b'']$, there exists an $h \in G$, such that

$$\|h^{(i)}(t) - \frac{d^i}{dt^i} S_\lambda(f,k,t)\|_{C[a,b]} \leq M'\ \lambda^{-(k+1)} \quad ,$$

$i = 0$ and $2k+2$. Therefore

$$K(\xi,f) \leq 3M'\ \lambda^{-(k+1)} + \|f(t)-S_\lambda(f,k,t)\|_{C[a,b]} +$$

$$\xi[\|S_\lambda(f,k,t)\|_{C[a,b]} + \|\frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(f,k,t)\|_{C[a,b]}].$$

Hence, it is sufficient to show that there exists an M, such that, for each $g \in G$

$$\|\frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(f,k,t)\|_{C[a,b]} \leq M\lambda^{k+1}\{\|f-g\| + \lambda^{-(k+1)}\|g^{(2k+2)}\|\}. \tag{4.11}$$

In fact, we have

$$\|\frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(f,k,t)\|_{C[a,b]}$$

$$\leq \sum_{j=0}^{k} |C(j,k)|\,\|\frac{d^{2k+2}}{dt^{2k+2}} \int W(d_j\lambda,t,u)[f(u)-g(u)]du\|_{C[a,b]} +$$

$$+ \sum_{j=0}^{k} |C(j,k)| \left\| \frac{d^{2k+2}}{dt^{2k+2}} \int W(d_j\lambda,t,u)g(u)\,du \right\|_{C[a,b]}$$

$$= I_1 + I_2 .$$

We first estimate I_1. From the relation $\frac{\partial}{\partial t} W(\lambda,t,u) = \frac{\lambda}{p(t)} W(\lambda,t,u)(u-t)$ [(1.8), (1.26) and (3.6)], we can easily obtain by induction that

$$(4.12) \quad \begin{cases} \dfrac{\partial^{2m}}{\partial t^{2m}} W(\lambda,t,u) = \displaystyle\sum_{i=0}^{m} \lambda^{m+i} W(\lambda,t,u)(u-t)^{2i} q_{i,2m}(u,t) \\[2ex] \dfrac{\partial^{2m+1}}{\partial t^{2m+1}} W(\lambda,t,u) = \displaystyle\sum_{i=0}^{m} \lambda^{m+i+1} W(\lambda,t,u)(u-t)^{2i+1} q_{i,2m+1}(u,t) + \\[2ex] \qquad\qquad + \lambda^{m} W(\lambda,t,u) q_{2m+1}(u,t) \end{cases}$$

where $q_{i,j}(u,t)$ are polynomials in u (and $\frac{1}{\lambda}$) which are bounded with respect to t for $t \in [a,b]$. Hence, we have [Lemmas II.1.8, II.2.8 and Corollary III.1.8]

$$(4.13) \quad \left\| \frac{d^{2k+2}}{dt^{2k+2}} \int W(d_j\lambda,t,u)(f(u)-g(u))\,du \right\|_{C[a,b]} \leq M_j \lambda^{k+1} \|f-g\|$$

(supp $f \cup$ supp $g \subset [a,b]$), where M_j is independent of g.

Secondly we estimate I_2. Estimating I_2 for Bernstein polynomials is easy, for by the formula (1.8) (1), p. 25 in [40], we have

$$(4.14) \qquad I_2 \leq \sum_{j=0}^{k} |C(j,k)|(1 - \frac{1}{d_j n}) \ldots (1 - \frac{2k+1}{d_j n}) \cdot$$

$$\cdot \int_0^1 W_3(d_j n-(2k+2), t; u) |g^{(2k+2)}(\eta_{u,t})| du$$

$$\leq \sum_{j=0}^{k} |C(j,k)| \, ||g^{(2k+2)}||_{C[a,b]} \; .$$

Similar estimates hold for Szasz and for Baskakov operators. However, to estimate I_2 for $S_n^2(f,t)$ we have to use a different approach.

Since $\lambda^i \int W(\lambda,t,u)u^i du$ is a polynomial in t and λ whose degree in t is i [Lemmas II.1.8, II.2.8 and III.1.8], it follows that

$$\frac{d^k}{dt^k} \int W(\lambda,t,u)u^i du = 0 \qquad \text{for} \quad k > i \; .$$

Therefore, as a linear combination of the above equations, we have

$$(4.15) \qquad \int \left[\frac{\partial^k}{\partial t^k} \quad W(\lambda,t,u) \right] (u-t)^i du = 0 \quad \text{for} \quad k > i \; .$$

Now for $g \in C_o^{2k+2}$, we first form

$$\frac{d^{2k+2}}{dt^{2k+2}} \int W(\lambda,t,u)g(u)du = \int \left[\frac{\partial^{2k+2}}{\partial t^{2k+2}} W(\lambda,t,u)\right]g(u)du \quad .$$

Then for the Taylor expansion,

$$g(u) = \sum_{\ell=0}^{2k+1} \frac{g^{\ell}(k)}{\ell!} (u+t)^{\ell} + \frac{g^{(2k+2)}(\xi)}{(2k+2)!} \quad (u-t)^{2k+2}$$

and use of (4.15), we have

(4.16) $$\left\|\frac{\partial^{2k+2}}{\partial t^{2k+2}} S_\lambda(g,t)\right\|_{C[a,b]}$$

$$\leq \frac{1}{(2k+2)!}\|g^{(2k+2)}\|\cdot\left\|\int\left|\left[\frac{\partial^{2k+2}}{\partial t^{2k+2}} W(\lambda,t,u)\right|(u-t)^{2k+2}du\right\|_{C[a,b]}.$$

Applying equation (4.12), and using the aforementioned lemmas, we can conclude that

(4.17) $$\left\|\frac{\partial^{2k+2}}{\partial t^{2k+2}} S_\lambda(g,t)\right\|_{C[a,b]} \leq M\|g^{(2k+2)}\|.$$

Hence $I_2 \leq M'\|g^{(2k+2)}\|$.

The proof of the lemma is now completed.

<u>Lemma IV.2.2</u>

Under the same assumption as in Lemma IV.2.1, there holds

(4.18) $$K(\xi,f) \leq M'\xi^{\alpha/2} \quad ,$$

i.e., $f \in C_o(\alpha,k+1;a',b')$.

<u>Proof</u>:

Let $L > 1$ be so large that $2M_0L^{-1+\frac{\alpha}{2}} \leq 1$, and $M_1 = \max\{K(1,f), 2M_0L^{\alpha/2}\}$, where M_0 is as in equation (4.10). Let $h_m = L^{-m}$. In the following we shall show by induction that

(4.19) $$K(h_m,f) \leq M_1\, h_m^{\alpha/2} \quad .$$

The relation (4.19) is obviously true for $m = 0$ (by the choice of M_1). Suppose that the inequality is true for $m-1$. Set $\xi = h_m$ and $\lambda^{-(k+1)} = h_{m-1}$ in the relation (4.10) of Lemma IV.3.2. Since $a \leq b+c$ implies either $a \leq 2b$ or $a \leq 2c$, it follows from (4.10) that either

$$(1) \quad K(h_m,f) \leq 2M_0 L^{\alpha/2} h_m^{\alpha/2} \leq M_1 h_m^{\alpha/2} ,$$

or

$$(2) \quad K(h_m,f) \leq 2M_0 (Lh_m)^{-1} h_m K(h_{m-1},f) .$$

But by induction (2) yields

$$\begin{aligned} K(h_m,f) &\leq 2M_0 L^{-1} M_1 h_{m-1}^{\alpha/2} \\ &= 2M_0 M_1 L^{-1+(\alpha/2)} h_m^{\alpha/2} \\ &\leq M_1 h_m^{\alpha/2} . \end{aligned}$$

So in both cases we have the inequality (4.19).

Now for any $\xi \in (0,1]$, we can find an m such that $h_m < \xi \leq h_{m-1}$. Since the K-function is increasing, it follows that

$$K(\xi,f) \leq K(h_{m-1},f) \leq M_1 L^{\alpha/2} h_m^{\alpha/2} \leq M' \,\alpha/2 ,$$

where $M' = M_1 L^{\alpha/2}$.

(Q.E.D.)

The previous two lemmas give one direction of the following theorem.

Theorem IV.2.3

Let $a < a' < a'' < b'' < a' < b$. If $f \in C_o$ with $\operatorname{supp} f \subset [a'',b'']$, then

$$||S_\lambda(f,k,t)-f(t)||_{C[a,b]} \leq M\,\lambda^{-\frac{\alpha}{2}(k+1)}$$

if and only if

$$f \in C_o(\alpha,k+1;a',b') \quad .$$

Proof:

It only remains to show that if $f \in C_o$ with $\operatorname{supp} f \subset [a'',b'']$ and $f \in C_o(\alpha,k+1;a',b')$, then

$$||S_\lambda(f,k,t)-f(t)||_{C[a,b]} \leq M\,\lambda^{-\frac{\alpha}{2}(k+1)} \quad .$$

To prove that, however, it is sufficient to show

$$(4.20) \qquad ||S_\lambda(f,k,t)-f(t)||_{C[a,b]} \leq M\,K(\lambda^{-(k+1)},f)$$

for some $M > 0$.

For $g \in G$ we have

$$(4.21) \quad ||S_\lambda(f,k,t)-f(t)||_{C[a,b]}$$

$$\leq \sum_{j=0}^{k} C(j,k)\int W(d_j\lambda,t,u)(f(u)-g(u))du||_{C[a,b]}$$

$$+ \sum_{j=0}^{k} C(j,k)\int W(d_j\lambda,t,u)g(u)du-f(t)||_{C[a,b]}$$

$$= I_1 + I_2 \; .$$

Clearly, $I_1 \leq M_1||f-g||$, since supp $(f-g) \subset [a,b]$.

In order to estimate I_2, we write

$$g(u) = g(t) + \sum_{m=1}^{2k+1} \frac{g^{(m)}(t)}{m!}(u-t)^m + \frac{1}{(2k+2)!} g^{(2k+2)}(\zeta)(u-t)^{2k+2}.$$

Using the properties of $C(j,k)$ [Lemma III.3.4] and the properties of $\int W(\lambda,t,u)(u-t)^m du$ [Lemmas II.1.8, II.2.8 and III.1.7], we can further calculate

$$(4.22) \qquad ||\sum_{j=0}^{k} C(j,k)\int W(d_j\lambda,t,u)g(u)du-f(t)||_{C[a,b]}$$

$$\leq ||g-f|| + M\lambda^{-(k+1)} \sum_{m=k+1}^{2k+2} ||g^{(m)}||$$

$$\leq ||g-f|| + M'\lambda^{-(k+1)}(||g|| + ||g^{(2k+2)}||)$$

where M is an absolute constant.

In other words

$$||S_\lambda(f,k,t)-f(t)||_{C[a,b]} \leq M'K(\lambda^{-(k+1)},f) ,$$

which completes the proof of the theorem.

Remark IV.2.4

It may be worthwhile pointing out the following fact:

In Thoerem IV.2.3, in order to conclude $f \in C_o(\alpha,k+1;a',b')$, it is enough to have a subsequence λ_n, $\lambda_n \uparrow \infty$ not faster than some geometric sequence, such that

$$||S_{\lambda_n}(f,k,t)-f(t)||_{C[a,b]} \le M\lambda_n^{-\frac{\alpha}{2}(k+1)} \quad .$$

In particular, the theorem is true for $\lambda_n = n$ as in the case of Bernstein polynomials.

This follows similarly to the result in Lemma IV.2.1 with λ being replaced by λ_n. That is,

$$(4.10') \qquad K(\xi,f) \le M_0[\lambda_n^{-\frac{\alpha}{2}(k+1)} + \lambda_n^{k+1}\,\xi K(\lambda_n^{-(k+1)},f)] \quad .$$

Since λ_n tends to infinitty and $\frac{\lambda_{n+1}}{\lambda_n} \le c$, c being a constant we can choose for any λ such that $\lambda_{n-1} < \lambda \le \lambda_n$. Then (4.10') yeilds the following relation:

$$(4.10'') \qquad K(\xi,f) \le M_0[\lambda^{-\frac{\alpha}{2}(k+1)} + c^{k+1}\lambda^{k+1}\,\xi K(\lambda^{-(k+1)},f)$$

$$\le M_0'\,[\lambda^{-\frac{\alpha}{2}(k+1)} + \lambda^{(k+1)}\xi\,K(\lambda^{-(k+1)},f)] \quad ,$$

which is the same expression as (4.10). The rest follows the same proof.

§3. Proof of Theorem IV.1.4 When $\operatorname{supp} f \subset (a,b)$

In this section we shall show that $f \in C_o(\alpha,k+1;a',b')$ is equivalent to $f \in Liz\,(\alpha,k+1;a,b)$ for f satisfying $\operatorname{supp} f \subset [a'',b'']$. In combination with the result of §2, this yields that

$$||S_\lambda(f,k,t)-f(t)||_{C[a,b]} = 0(\lambda^{-\frac{\alpha}{2}(k+1)})$$

is equivalent to the condition $f \in \text{Liz}\,(\alpha,k+1;a,b)$ whenever supp $f \in [a'',b'']$.

Theorem IV.3.1

Let $a < a' < a'' < b'' < b' < b$. If $f \in C_o$ with supp $f \subset [a'',b'']$, then

$$f \in C_o(\alpha,k+1;a',b') \text{ if and only if } f \in \text{Liz}\,(\alpha,k+1;a,b).$$

Proof.

Assuming $f \in C_o(\alpha,k+1;a',b')$, we shall show that

$$\omega_{2k+2}(f,h) \equiv \omega_{2k+2}(f,h,a,b) = 0(h^{\alpha(k+1)}) \quad .$$

Let $h > 0$ and $|\delta| < h$. Then for any $g \in G$, we have

$$(4.23) \qquad |\Delta_\delta^{2k+2} f(t)|$$

$$\leq \sum_{i=0}^{2k+2} \binom{2k+2}{i} |f(t+i\delta) - g(t+i\delta)| + |\Delta_\delta^{2k+2} g(t)|$$

$$\leq 2^{2k+2} ||f-g|| + \delta^{2k+2} |g^{(2k+2)}(t+\eta)|$$

$$\leq 2^{2k+2} ||f-g|| + \delta^{2k+2} ||g^{(2k+2)}|| \, .$$

Thus,

$$(4.24) \qquad |\Delta_\delta^{2k+2} f(t)| \leq 2^{2k+2} K(\delta^{2k+2},f)$$

$$\leq 2^{2k+2} M |\delta|^{\alpha(k+1)}$$

$$\leq 2^{2k+2} M h^{\alpha(k+1)} \quad .$$

Hence, $\omega_{2k+2}(f,h) \leq 2^{2k+2} M h^{\alpha(k+1)}$.

On the other hand, assume $f \in \mathrm{Liz}(\alpha,k+1;a,b)$. In order to show that $f \in C_0(\alpha,k+1;a',b')$, it is enough to show $K(\xi,f) \leq M\xi^{\alpha/2}$ when ξ is sufficiently small.

Define $g_0 \in G$ by

$$(4.25) \qquad g_0(x) = \frac{1}{\binom{2k+2}{k+1}\eta^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} [(-1)^k \bar{\Delta}^{2k+2}_{\sum_{i=1}^{2k+2} U_i} f(x) + \binom{2k+2}{k+1} f(x)]\, du_1 \ldots du_{2k+2}$$

where $(k+1)\eta < \min(a''-a', b'-b'', k+1)$ and $\bar{\Delta}_h^m$ is the symmetric difference. It follows that $\operatorname{supp} g_0 \subset [a',b']$.

We shall show in the remainder of this section the following relation:

$$(4.26) \qquad K(\eta^{2k+2},f) \leq M\eta^{(k+1)} \quad \text{for some} \quad M > 0,$$

which would complete the proof of the sufficiency of Theorem IV.4.1 (for functions of compact supports).

First observe that, from the definition of g_0, we have

$$(4.27) \qquad |f(x)-g_0(x)| \leq \frac{1}{\binom{2k+2}{k+1}\eta^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} |\bar{\Delta}_{\Sigma u_i} f(x)| \cdot du_1 \ldots du_{2k+2}$$

$$\leq M\, \omega_{2k+2}(f,\eta(k+1))$$

$$\leq M'\, \eta^{\alpha(k+1)} .$$

or

$$||f-g|| \leq M' \; \eta^{\alpha(k+1)} \quad .$$

Also, it is clear that $||g_0|| \leq \frac{(2^{2k+2}-1)}{\binom{2k+2}{k+1}} ||f||$.

On the other hand, we claim that

$$(4.28) \qquad ||g_0^{(2k+2)}|| \leq M \, \eta^{-(2k+2)} \; \omega_{2k+2}(f,\eta) \quad ,$$

where M is a fixed constant.

This would be enough for proving (4.26), and therefore for completing the proof of the theorem since equation (4.28) would imply that $||g_0^{(2k+2)}|| \leq M'\eta^{-(2k+2)+(k+1)}$, and hence

$$\begin{aligned} K(\eta^{2k+2},f) &\leq ||f-g_0|| + \eta^{2k+2}[||g_0|| + ||g_0^{2k+2}||] \\ &\leq M \, \eta^{(k+1)\alpha} \quad . \end{aligned}$$

In order to show (4.28), notice that

$$(4.29) \qquad (-1)^k \binom{2k+2}{k+1} \eta^{2k+2} \; g_0(x)$$

$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \{\bar{\Delta}^{2k+2}_{\Sigma_{j=1}^{2k+2} u_j} f(x) + (-1)^k \binom{2k+2}{k+1} f(x)\} \cdot$$

$$\cdot \; du_1 \ldots du_{2k+2}$$

$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[\sum_{i=0}^{2k+2} (-1)^i \binom{2k+2}{i} f\{x+(k+1-i) \sum_{j=1}^{2k+2} u_j\} + \right.$$

$$\left. + \; (-1)^k \binom{2k+2}{k+1} f(x) \right] du_1 \ldots du_{2k+2} \quad .$$

Further, observe that

$$(4.30)\qquad \frac{d^{2k+2}}{dx^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[f\left(x + \sum_{i=1}^{2k+2} u_i\right) + f\left(x - \sum_{i=1}^{2k+2} u_i\right)\right] \cdot du_1 \ldots du_{2k+2} = 2\,\bar{\Delta}^{2k+2} f(x) \quad ,$$

and

$$(4.31)\qquad \omega_{2k+2}(f,(k+1-i)\eta) \leq (k+1-i)\,\omega_{2k+2}(f,\eta) \quad .$$

Using (4.29), (4.30) and (4.31),

$$\|g_0^{(2k+2)}\|$$

$$= \frac{\eta^{-(2k+2)}}{\binom{2k+2}{k+1}} \left\| \frac{d^{2k+2}}{dx^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \sum_{i=0}^{k} (-1)^i \binom{2k+2}{i} \cdot \left[f\{x+(k+1-i) \sum_j u_j\} + f\{x-(k+1-i) \sum_j u_j\}\right] du_1 \ldots du_{2k+2} \right\|$$

$$= \frac{\eta^{-(2k+2)}}{\binom{2k+2}{k+1}} \left\| \sum_{i=0}^{k} (-1)^i \binom{2k+2}{i} \cdot 2\bar{\Delta}^{2k+2}_{(k+1-i)\eta} f(x) \right\|$$

$$\leq \eta^{-(2k+2)} \frac{2}{\binom{2k+2}{k+1}} \sum_{i=0}^{k} \binom{2k+2}{i} (k+1-i)\,\omega_{2k+2}(f,\eta)$$

which proves (4.28).

(Q.E.D.)

§4. Proof of Theorem IV.1.4. The General Case

In this section, we shall prove the inverse theorem for the general case. The proof will be divided into two parts: The implication (2) to (4) and the implication (1) to (3). The equivalence of (2) and (3) is known.

Before the proof, we first observe the following fact, which will be frequently used in the proof.

<u>Lemma IV.4.1</u>

Let $[c,d] \subset (a,b)$, $|f(t)| \leq M\,\psi_A(t)$. Then for any $m > 0$,

$$(4.32) \qquad \Big\| \int_{u \notin (a,b)} w(\lambda,t,u) f(u)\,du \Big\|_{C[c,d]} = o(\lambda^{-m})$$

($\psi_A(t)$ was given in Definition IV.1.3 (1°)).

Lemma IV.4.1 is in fact a direct consequence of Lemmas II.1.8, II.2.8 and III.1.8. In fact, for $\delta = \min\{c-a, b-d\}$, we have

$$\Big\| \int_{u \notin (a,b)} w(\lambda,t,u) f(u)\,du \Big\|_{C[c,d]}$$

$$\leq M \Big\| \int W(\lambda,t,u)\,\psi_A^2(u)\,du \Big\|_{C[a,b]}^{1/2} \Big\| \frac{1}{\delta^{4m+2}} \int W(\lambda,t,u)(u-t)^{4m+2}\,du \Big\|_{C[c,d]}^{1/2}$$

$$= \quad o(\lambda^{-m})$$

by the Cauchy-Schwarz inequality.

§4.1 <u>The Implication (2) ⟹ (4)</u>

Assuming $f \in \mathrm{Liz}\,(\alpha,k+1;a_2,b_2)$, we shall show

$$\|S_\lambda(f,k,t)-f(t)\|_{C[a_3,b_3]} = 0(\lambda^{-\frac{\alpha}{2}(k+1)}) \quad .$$

Choose a', a'', b', b'' in such a way that $a_2 < a' < a'' < a_3 < b_3 < b'' < b' < b_2$. Let $g \in C_o^\infty$ be such that $g(x) = 1$ for $x \in [a'',b'']$, and $\operatorname{supp} g \subset [a',b']$. Then fg has compace support strictly in (a_2,b_2), and $fg \in \mathrm{Liz}\,(\alpha,k+1;a_2,b_2)$ since f does. Hence, by Theorems IV.2.3 and IV.3.1,

$$\|S_\lambda(fg,k,t)-fg(t)\|_{C[a_2,b_2]} = 0(\lambda^{-\frac{\alpha}{2}(k+1)}) \quad .$$

But, for $t \in [a_3,b_3]$,

(4.33) $$S_\lambda(fg,k,t)-f(t)g(t)$$

$$= \sum_{j=0}^{k} C(j,k) \int W(d_j\lambda,t,u)(f(u)g(u)-f(t)g(t))du$$

$$= \sum_{j=0}^{k} C(j,k) \int_{a''}^{b''} W(d_j\lambda,t,u)(f(u)g(u)-f(t)g(t))du+o(\lambda^{-(k+1)})$$

where the remainder $o(\lambda^{-(k+1)})$ is uniform for $t \in [a_3,b_3]$ (Lemma IV.4.1). Noticing that $g(u) = 1$ on $[a'',b''] \supset [a_3,b_3]$ we have further

(4.34) $$S_\lambda(fg,k,t)-f(t)g(t)$$

$$= \sum_{j=0}^{k} C(j,k) \int_{a''}^{b''} W(d_j\lambda,t,u)(f(u)-f(t))du + o(\lambda^{-(k+1)})$$

$$= \sum_{j=0}^{k} C(j,k) \int W(d_j\lambda,t,u)(f(u)-f(t))du + o(\lambda^{-(k+1)})$$

$$= S_\lambda(f,k,t)-f(t) + o(\lambda^{-(k+1)}) \quad ,$$

where the little "o" terms are uniform for $t \in [a_3,b_3]$ by Lemma IV.4.1. Consequently,

$$||S_\lambda(f,k,t)-f(t)||_{C[a_3,b_3]} = O(\lambda^{-\frac{\alpha}{2}(k+1)}) \quad .$$

§4.2 The Implication (1) ⟹ (3).

Now we assume

$$||S_\lambda(f,k,t)-f(t)||_{C[a_1,b_1]} = O(\lambda^{-\frac{\alpha}{2}(k+1)}) \quad ,$$

and prove condition (3) in the theorem.

We shall prove it by "induction" on $\tau \equiv \alpha(k+1)$. The induction proceeds as follows:

First, we prove it for the case when $0 < \tau \leq 1$. Then, for any $\delta \in (0,1)$, we prove the case when $1-\delta < \tau < 2-\delta$. In general, we assume the proposition holds for $0 < \tau \leq m-\delta$, $m = 1,2,\ldots,2k+1$, $0 < \delta < \frac{1}{2}$, and then prove the case when $m-\delta \leq \tau < m+1-2\delta$. Since $\delta > 0$ can be chosen arbitrarily small, the proposition will hold for all $\tau \in (0,2k+2)$.

§4.2.1 The Case $0 < \tau \leq 1$.

Let a', a'', b', b'' be chosen so that $a_1 < a' < a'' < a_2$

and $b_2 < b'' < b' < b_1$. Also, let $g \in C_o^\infty$ be such that $\operatorname{supp} g \subset [a'',b'']$ and $g(x) = 1$ on $[a_2,b_2]$.

Lemma IV.4.2

Let g be chosen as above. If

$$||S_\lambda(f,k,t)-f(t)||_{C[a_1,b_1]} = O(\lambda^{-\tau/2}), \quad 0 < \tau \leq 1 ,$$

then

$$||S_\lambda(fg,k,t)-fg(t)||_{C[a',b']} = O(\lambda^{-\tau/2}) \quad .$$

Proof.

For $t \in [a',b']$, we have

$$(4.35) \qquad S_\lambda(fg,k,t)-f(t)g(t)$$

$$= \sum_{j=0}^{k} C(j,t) \int_{a_1}^{b_1} W(d_j\lambda,t,u)(f(u)g(u)-f(t)g(t))du + + o(\lambda^{-(k+1)})$$

$$= g(t) \sum_{j=0}^{k} C(j,t) \int W(d_j\lambda,t,u)(f(u)-f(t))du$$

$$+ \sum_{j=0}^{k} C(j,k) \int_{a_1}^{b_1} W(d_j\lambda,t,u)f(u)(g(u)-g(t))du$$

$$+ o(\lambda^{-(k+1)})$$

$$\equiv I_1(t) + I_2(t) + o(\lambda^{-(k+1)})$$

where the little "o" terms in (4.35) are uniform for $t \in [a',b']$ (Lemma IV.4.1).

The assumption

$$||S_\lambda(f,k,t)-f(t)||_{C[a_1,b_1]} = O(\lambda^{-\frac{\alpha}{2}(k+1)})$$

yields the estimate

$$(4.36) \qquad ||I_1(t)||_{C[a',b']} \leq ||g|| \cdot ||S_\lambda(f,k,t)-f(t)||_{C[a',b']}$$
$$\leq M_1 \lambda^{-\tau/2} \quad .$$

Next, by the mean value theorem, we reduce $I_2(t)$ to

$$(4.37) \qquad I_2(t) = \sum_{j=0}^{k} C(j,k) \int_{a_1}^{b_1} W(d_j\lambda,t,u)f(u)\{g'(\xi)(u-t)\}du \quad .$$

Hence,

$$(4.38) \qquad ||I_2(t)||_{C[a',b']} \leq \sum_{j=0}^{k} |C(j,k)|\{||g'|| \cdot ||\int_{a_1}^{b_1} W(d_j\lambda,t,u) \cdot$$
$$\cdot |f(u)| \cdot |u-t| du||_{C[a',b']}\}$$
$$\leq ||f||_{C[a',b']} ||g'|| \cdot \left(\sum_{j=0}^{k} |C(j,k)| \right) \cdot$$
$$\cdot \max_{0 \leq j \leq k} ||\int W(d_j\lambda,t,u)|u-t|du||_{C[a',b']} \; .$$

Using the Cauchy-Schwarz inequality, we obtain

$$(4.39) \quad \|I_2(t)\|_{C[a',b']} \le \|f\|_{C[a',b']} \|g'\| \left(\sum_{j=0}^{k} |C(j,k)| \right) \cdot$$

$$\cdot \max_{o\le j\le k} \left\| \int W(d_j\lambda,t,u)(u-t)^2 du \right\|^{1/2}_{C[a'b']}$$

$$\le \|f\|_{C[a'b']} \|g'\| \left(\sum_{j=0}^{k} |C(j,k)| \right) \frac{\|p(t)\|^{1/2}_{C[a'',b'']}}{\lambda^{1/2}}$$

$$= O(\lambda^{-1/2}) = O(\lambda^{-\tau/2}) \quad .$$

Combining (4.35), (4.36) and (4.39), we conclude that

$$\|S_\lambda(fg,k,t)-fg(t)\|_{C[a',b']} = O(\lambda^{-\tau/2}) \quad .$$

Lemma IV.4.3

Let $a_1 < a_2 < b_2 < b_1$. If

$$\|S_\lambda(f,k,t)-f(t)\|_{C[a_1,b_1]} = O(\lambda^{-\tau/2}) ,$$

$0 < \tau < 1$, then

$$(4.40) \quad \begin{cases} f \in \mathrm{Lip}\,(\tau,a_2,b_2) & \text{if} \quad \tau < 1 \\ f \in \mathrm{Lip}^*(1,a_2,b_2) & \text{if} \quad \tau = 1 \end{cases}$$

Proof.

Let a', a'', b', b'' and g be chosen as above. Since supp $fg \subset [a'',b''] \subset (a',b')$, it follows from Theorems IV.2.3, IV.3.1 and Lemma IV.4.2, that

$$(4.41)\qquad \begin{cases} fg \in \mathrm{Lip}\,(\tau,a',b') & \text{if} \quad \tau < 1 \\ fg \in \mathrm{Lip}^*(1,b',b') & \text{if} \quad \tau = 1 \end{cases} .$$

Noticing that $g(t) = 1$ for $t \in [a_2,b_2]$, equation (4.40) is deduced from (4.41).

§4.2.2 The Induction Process

Assume that the proposition holds for $0 < \tau \leq m-\delta$ ($m = 1,2,\ldots,2k+1$; $0 < \delta < \frac{1}{2}$) and suppose that

$$(4.42)\qquad \|S_\lambda(f,k,t)-f(t)\|_{C[a,b]} \leq M_\tau \lambda^{-\tau/2}$$

where $m-\delta \leq \tau \leq m+1-2\delta$, we must deduce (3) of Theorem IV.1.4 for that τ.

Let x_i, y_i, $i = 1,2,3$ be chosen such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1$. Let $g \in C_o^\infty$ with $\operatorname{supp} g \subset (x_3,y_3)$ and $g(x) = 1$ on $[a_2,b_2]$.

<u>Lemma IV.4.4</u>

Let g be chosen as above. If

$$\|S_\lambda(f,k,t)-f(t)\|_{C[a_1,b_1]} = 0(\lambda^{-\tau/2}) \quad ,$$

then

$$\|S_\lambda(fg,k,t)-fg(t)\|_{C[x_2,y_2]} = 0(\lambda^{-\tau/2}) \quad ,$$

where $m-\delta \leq \tau \leq m+1-2\delta$.

First notice that, by induction hypothesis, relation (4.42) with $\tau = m-\delta$ implies that $f^{(m-1)}$ exists and

$f^{(m-1)} \in \mathrm{Lip}\,(1-\delta, x_1, y_1)$. Next, for $t \in [x_2, y_2]$, we form

$$(4.43) \qquad S_\lambda(fg,k,t) - f(t)g(t)$$

$$= \sum_{j=0}^{k} C(j,k) \int W(d_j\lambda,t,u) f(u)g(u)\,du - f(t)g(t)$$

$$= \sum_{j=0}^{k} C(j,k) \int_{x_1}^{y_1} W(d_j\lambda,t,u)(f(u)-f(t))(g(u)-g(t))\,du$$

$$+ \sum_{j=0}^{k} C(j,k) \int W(d_j\lambda,t,u)(g(u)-g(t))\,du \cdot f(t)$$

$$+ \sum_{j=0}^{k} C(j,k) \int W(d_j\lambda,t,u)(f(u)-f(t))\,du \cdot g(t)$$

$$+ \ o(\lambda^{-(k+1)})$$

$$\equiv I_1 + I_2 + I_3 + o(\lambda^{-k-1})$$

where the $o(\lambda^{-k-1})$ term is uniform for $t \in [x_2, y_2]$ (Lemma IV.4.1).

The estimates for I_2 and I_3 can be made immediately:

$$(4.44) \qquad \|I_3\|_{C[x_2,y_2]} \leq \|g\| \cdot \|S_\lambda(f,k,t) - f(t)\|_{C[x_2,y_2]}$$

$$= O(\lambda^{-\tau/2})$$

by the assumption of the lemma (and $[x_2, y_2] \subset (a_1, b_1)$); and

$$(4.45) \qquad \|I_2\|_{C[x_2,y_2]} \leq \|f\|_{C[x_2,y_2]} \|S_\lambda(g,k,t) - g(t)\|_{C[x_2,y_2]}$$

$$= O(\lambda^{-(k+1)/2}) = O(\lambda^{-\tau/2})$$

as $g \in C_o^\infty \subset C^{2k+2}$.

In the estimating of $||I_1||_{C[x_2,y_2]}$, we see that by induction hypothesis, $f^{(m-1)}$ exists on $[x_1,y_1]$. so that, for $t \in [x_2,y_2]$, by Taylor's expansion,

$$(4.46) \qquad \sum_{j=0}^{k} C(j,k) \int_{x_1}^{y_1} W(d_j\lambda,t,u)(f(u)-f(t))(g(u)-g(t))du$$

$$= \sum_{i=1}^{m-1} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{k} C(j,k) \int_{x_1}^{y_1} W(d_j\lambda,t,u)(u-t)^i(g(u)-g(t))du$$

$$+ \frac{1}{(m-1)!} \sum_{j=0}^{k} C(j,k) \int_{x_1}^{y_1} W(d_j\lambda,t,u)(u-t)^{m-1}(f^{(m-1)}(\xi) -$$

$$- f^{(m-1)}(t)) \cdot g'(\eta)(u-t)du$$

$$\equiv I_4 + I_5$$

where ξ and η are between u and t.

Clearly,

$$||I_4||_{C[x_2,y_2]} = O(\lambda^{-(k+1)/2}) = O(\lambda^{-\tau/2}) \quad .$$

Also, since $f^{(m-1)} \in \text{Lip}\ (1-\delta,x_1,y_1)$,

$$(4.47) \qquad |f^{(m-1)}(\xi)-f^{(m-1)}(t)| \le M|\xi-t|^{1-\delta} \le M|u-t|^{1-\delta}$$

for some $M > 0$. Therefore,

$$||I_5||_{C[x_2,y_2]}$$

$$\le M||g'|| \cdot \frac{\sum_{j=0}^{k} |C(j,k)|}{(m-1)!} \max_{0\le j\le k} ||\int_{x_1}^{y_1} W(d_j\lambda,t,u) \cdot$$

$$\cdot |u-t|^{m+1-\delta} du||_{C[x_2,y_2]}$$

$$\leq M' \max_{0\leq j\leq k} \left\| \int_{x_1}^{y_1} W(d_j\lambda,t,u)|u-t|^{2(m+1)}du \right\|_{C[x_2,y_2]}^{(m+1-\delta)/2(m+1)} ,$$

by Jensen's inequality. Thus

$$(4.49) \qquad \|I_5\|_{C[x_2,y_2]} = O(\lambda^{-(m+1-\delta)/2})$$

$$= O(\lambda^{-\tau/2})$$

by Lemmas II.1.8, II.2.8 and III.1.8.

Combining the above estimates, we obtain

$$\|S_\lambda(fg,k,t)-fg(t)\|_{C[x_2,y_2]} = O(\lambda^{-\tau/2}) \quad .$$

Lemma IV.4.5

Let $a_1 < a_2 < b_2 < b_1$. If

$$\|S_\lambda(f,k,t)-f(t)\|_{C[a_1,b_1]} = O(\lambda^{-\tau/2})$$

$m-\delta < \tau < m+1-2\delta$, then

$$(4.50) \quad \begin{cases} (1) \quad f^{(m-1)} \text{ exists, } f^{(m-1)} \in \text{Lip}\,(\tau-m+1,a_2,b_2)\,, \\ \qquad \text{if } m-\delta < \tau < m \\ (2) \quad f^{(m-1)} \text{ exists, } f^{(m-1)} \in \text{Lip}^*(1,a_2,b_2), \text{ if } \tau = m \\ (3) \quad f^{(m)} \text{ exists, } f^{(m)} \in \text{Lip}\,(\tau-m,a_2,b_2), \\ \qquad \text{if } m < \tau < m+1-2\delta. \end{cases}$$

Proof.

Let x_i, y_i, $i = 1,2,3$ and g be defined as in

Lemma IV.4.4. As an intermediate result, Lemma IV.4.4 yields

$$||S_\lambda(fg,k,t)-fg(t)||_{C[x_2,y_2]} = O(\lambda^{-\tau/2}) \quad .$$

Furthermore, since fg has compact support in $[x_3,y_3] \subset (x_2,y_2)$, Theorems IV.2.3 and IV.3.1 imply (by virtue of the equivalence (2) $\Longleftrightarrow$ (3)) that

(i) $(fg)^{(m-1)}$ exists and $(fg)^{(m-1)} \in \mathrm{Lip}\,(\tau-m+1,a_2,b_2)$, if $m-\delta < \tau < m$;

(ii) $(fg)^{(m-1)}$ exists $(fg)^{(m-1)} \in \mathrm{Lip}^*(1,a_2,b_2)$, if $\tau = m$;

(iii) $(fg)^{(m)}$ exists and $(fg)^{(m)} \in \mathrm{Lip}\,(\tau-m,a_2,b_2)$, if $m < \tau < m+1-2\delta$.

Since $g(x) = 1$ on $[a_2,b_2]$, restricting ourselves to this interval, we can replace fg by f in (i), (ii) and (iii) above, which gives the equation (4.50).

(Q.E.D.)

§5 An Application to the Saturation Problem of the General Combinations

As we have stated in Chapter III, Section 2, an interesting application to the optimal case of approximation can be derived from the non-optimal results we have just proved.

Recall that, in the proofs of the saturation theorems in Chapters II and III, we needed an intermediate result, that is

the existence of $f^{(2k)}$, to ensure the interchange of limits of expressions of the form:

$$(4.51)\qquad \lim_{\ell \to \infty} \lim_{\lambda_j \to \infty} < \lambda_j^{k+1}[S_{2\lambda_j}(f_\ell,k,\cdot)-S_\lambda(f_\ell,k,\cdot),g(\cdot)> ,$$

which have been proved by induction on k in the corresponding theorems.

We have pointed out that we are not able to carry the induction procedures when $S^i_\lambda(f,k,t)$ is representing the general combinations defined in (4.1) and (4.2). Therefore, in order to show the corresponding saturation theorems also hold for such general combinations $S_\lambda(f,k,t)$, we have to justify the interchanging of the limits in (4.51). The interchanging of the limits would be permitted if $f^{(2k)} \in C_{[a'',b'']}$, $(a'',b'') \supset \text{supp } g$, where g is the corresponding function in (4.51). However, this is a direct consequence of the inverse theorem. (In fact, we can conclude from the inverse theorem that $f^{(2k+1)} \in \text{Lip } (\alpha;a'',b'')$ for any $\alpha \in (0,1)$.)

Moreover, a slightly stronger result can be derived: instead of using the subsequence $\lambda_i = 2^i\lambda_o$ in the statement of the saturation theorems (i.e., Theorems II.1.2, II.2.1 and III.1.2), one can use correspondingly any subsequence λ_i of λ such that $\frac{\lambda_{i+1}}{\lambda_i} \leq M$.

It is worthwhile noting that the saturation theorem for "general" combinations holds for Phillips operators too, since in the proof of the saturation theorem II.1.2 the induction step was not used.

We conclude this section with the following theorem:

Theorem IV.5.1

Suppose $0 < a < a_1 < b_1 < b \in \Omega$, f satisfies condition A (defined in Definition IV.1.3), and λ_j is a sequence tending to infinity not faster than some geometric sequence, then for $S_\lambda(f,k,t)$ (denoting general combinations defined in (4.1) and (4.2)) of a Bernstein-type operator and for

$$I(f,\lambda,k,a,b) \equiv \lambda^{k+1} \| S_\lambda(f,k,\cdot)-f(\cdot)\|_{C[a,b]} ,$$

the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) hold, where (i) $i = 1,\dots,6$ are given by:

(1) $I(f,\lambda_j,k,a,b) = O(1), \quad \lambda_j \to \infty$;

(2) $f^{(2k+1)} \in A.C.(a,b)$ and $f^{(2k+2)} \in L_\infty[a,b]$;

(3) $I(f,\lambda,k,a_1,b_1) = O(1), \quad \lambda \to \infty$;

(4) $I(f,\lambda_j,k,a,b) = o(1), \quad \lambda_j \to \infty$;

(5) $f \in C^{2k+2}(a,b)$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t) f^{(i)}(t) = 0$,

$t \in (a,b)$, where $Q(i,k,t)$ are polynomials depending on k;

(6) $I(f,\lambda,k,a_1,b_1) = o(1), \quad \lambda \to \infty$.

CHAPTER V

EXPONENTIAL FORMULAE FOR SEMIGROUPS OF OPERATORS

§1 Semigroups: Definitions and Remarks

Saturation and inverse results for the Bernstein-type operators can be further generalized as we shall discuss in this chapter. Applications to exponential formulae for semigroups of operators will be given.

We begin by stating some definitions. For a detailed discussion of semigroups, we refer the reader to the book of P.L. Butzer and H. Berens [10].

<u>Definition V.1.1</u>

1° A family of operators $\{T(t);\ t \geq 0\}$ defined on a Banach space X is called a C_o-semigroup of operators if the following conditions are satisfied:

(i) $T(0) = I$, the identity operator;

(ii) $T(t_1+t_2) = T(t_1)T(t_2)$;

(iii) $\underset{t \to 0+}{\text{s-lim}}\ T(t)f = f, \quad (f \in X).$

2° The operator A on (a subspace of) X defined by

$$Af = \underset{\tau \to 0}{\text{s-lim}} \frac{T(\tau)f-f}{\tau} \quad .$$

whenever the limit exists, is called the infinitesimal generator of $\{T(t)\}$.

The operators A^r, $r = 0,1,2,\ldots$, are defined inductively by the relations $A^0 = I$, $A^1 = A$ and $A^r f = A(A^{r-1}f)$, $r = 1,2,3,\ldots$.

The subspace $D(A^r)$ of X denotes the domain of A^r, $r = 0,1,2,\ldots$.

§2 Exponential Formulae

A C_o semigroup of operators $\{T(t)\}$ can be approximated by formulae known as exponential formulae ([28], p. 359). The rates of convergence of some of the exponential formulae in terms of the moduli of continuity of $T(t)f$ and $AT(t)f$ were investigated in [10], [16] and [17]. A saturation theorem for some of the exponential formulae can be found in [22]. In [22], we have investigated the exponential formulae of Hille, Kendall, Post-Widder and Phillips.

The aim of this chapter is to investigate the saturation and inverse problems for certain linear combinations involving these exponential formulae.

For $\{T(u)\}$ a C_o-semigroup of operators on a Banach space X, we define:

<u>Definition V.2.1</u>

The exponential formulae are the operators $H_i(\lambda,k,t)$ on X given by:

$$(5.1)\qquad H_i(\lambda,k,t)f(\cdot) = \sum_{j=0}^{k} C(j,k) \int_0^\infty W_i(\lambda,k,t)T(u)f(\cdot)du$$

where

$$W_1(\lambda,t,u) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta(u - \frac{k}{\lambda}) , \quad t > 0 ;$$

$$W_2(n,t,u) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \delta(u - \frac{k}{n}) , \quad 0 < t < 1 ;$$

$$W_3(n,t,u) = \frac{1}{(n-1)!} (\frac{n}{t})^n e^{-nu/t} u^{n-1} , \quad t > 0 ;$$

$$W_4(\lambda,t,u) = \sqrt{\lambda/2\pi}\, e^{-(u-t)^2\lambda/2} , \quad t > 0 ;$$

$$W_5(\lambda,t,u) = e^{-\lambda(t+u)} \left[\sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n u^{n-1}}{n!\,(n-1)!} + \delta(u) \right] , \quad t > 0 ;$$

and

$$C(j,k) = \prod_{\substack{i \neq j \\ i=0}}^{k} \frac{d_j}{d_j - d_i}$$

for distinct positive integers $d_0, d_1, \ldots, d_k$.

$H_i(\lambda,0,t)$, $i = 1,2,3$ and 5 are usually called the exponential formulae for semigroups of operators of Szasz, Kendall, Post-Widder and Phillips, respectively.

Definition V.2.2

The subspace $B_{2k+2}(t_o)$ of X is defined as

$$(5.2) \qquad B_{2k+2}(t_o) = \{f \in X;\ T(t)f \in D(A^{2k+2}),\ t > t_o\} .$$

Definition V.2.3

The subspaces $\mathrm{Lip}_T(\alpha;t_o)$, $\mathrm{Lip}_T^*(\alpha;t_o)$ and

$\mathrm{Liz}_T\ (\alpha,k,t_o)$ $(k = 0,1,2,\ldots)$ are given by:

(5.3) $$\mathrm{Lip}_T\ (\alpha;t_o) \equiv \{f \in X;\ \omega_T^1(f,h;\xi) = O(h^\alpha),\ \xi > t_o\}\ ;$$

$$\mathrm{Lip}_T^*\ (\alpha;t_o) \equiv \{f \in X;\ \omega_T^2(f,h;\xi) - O(h^\alpha),\ \xi > t_o\}\ ;$$

$$\mathrm{Liz}_T\ (\alpha,k;t_o) \equiv \{f \in X;\ \omega_T^{2k}(f,h;\xi) = O(h^{\alpha k}),\ \xi > t_o\}\ ,$$

where

$$\omega_T^k(f,h;\xi) \equiv \sup\{\|\Delta_\delta^k\ T(t)f\|_X;\ |\delta| \leq h,\ \xi \leq t \leq \xi+1\}\ .$$

§3 Saturation and Inverse Results for Exponential Formulae

From the similarities of the expressions for the exponential formulae defined in Definition V.2.1 and the classical operators discussed in the previous chapters, one would expect that similar saturation and inverse results would be proved for exponential formulae for semi-groups of operators.

As a matter of fact, the following theorems can be reduced easily from our previous results.

Theorem V.3.1

Let $f \in X$, and $\{T(t);\ t \geq 0\}$ be a C_o-semigroup on X, where X is a reflexive Banach space. Then for any $i = 1,2,\ldots,5$, and $t_o > 0$, the following are true:

(1) $\lambda^{k+1}\|H_i(\lambda,k,t)f-T(t)f\|_{C([a,b],X)} \leq M(a,b)$ for all $[a,b] \subset (t_o,t_o+\delta)$, if and only if $f \in B_{2k+2}(t_o)$;

(2) $\lambda^{k+1}\|H_i(\lambda,k,t)f-T(t)f\|_{C([a,b],X)} = o(1)$ for all $[a,b] \subset (t_o,t_o+\delta)$, if and only if $f \in D(A^{2k+2}T(t_o))$ and $\sum_{j=k+1}^{2k+2} Q_i(j,k,t_o)A^j\, T(t_o)f = 0$, where $Q_i(j,k,f)$ are polynomials depending on k and i.

Theorem V.3.2

If $\{T(t);\ t \geq 0\}$ is a C_o-semigroup on a reflexive Banach space X, $f \in X$, $t_o > 0$ and $0 < \alpha < 2$, then, for $i = 1, 2, 3$ and 4, we have

$$\lambda^{\frac{\alpha}{2}(k+1)}\|H_i(\lambda,k,t)-T(t)f\|_{C([a,b],X)} \leq M(a,b)$$

for all $[a,b] \subset (t_o,t_o+\delta)$, if and only if $f \in Liz_T\,(\alpha,k+1;t_o)$.

§4 The Saturation Theorem

Theorems V.3.1 and V.3.2 are consequences of the theorems proved in the previous chapters. To see this, we need a slightly more general version for those theorems.

We shall examine this problem more closely. For convenience, we shall not state the change in the conditions in case of Baskakov operators, since we do not need it in the formulae for semigroups of operators.

Suppose $f \in C([0,\infty),X)$, X is a reflexive Banach space, $\|f(t)\|_X \leq Me^{Nt}$ then $S_\lambda(f,k,t)$, $k = 0,1,2,\ldots$ are functions in $C([0,\infty),X)$, which converge to $f(t)$ pointwise with respect

to the norm in X.

To prove the saturation problems, we use first arguments similar to those of Lemmas II.1.3 and II.2.3 to show the equivalence of

$$\lambda_n^{k+1} \|S_{\lambda_n}(f,k,t)-f(t)\|_{C([a,b],X)} = 0(1)$$

and

$$\lambda_n^{k+1} \|S_{2\lambda_n}(f,k,t)-S_{\lambda_n}(f,k,t)\|_{C([a,b]} \quad .$$

Then, by Alaoglu's theorem, there exists a function $h \in L_\infty([a,b],X)$ and a subsequence $\{\lambda_{n_\rho}\}$ of $\{\lambda_n\}$ such that for any $g \in C_o^\infty([0,\infty),X^*)$, X^* is the dual space of X, $\operatorname{supp} g \subset (a,b)$, we have

$$(5.4) \qquad \langle\lambda_{n_\rho}^{k+1} [S_{2\lambda_{n_\rho}}(f,k,\cdot)-S_{\lambda_{n_\rho}}(f,k,\cdot)],g(\cdot)\rangle \to \langle h(.),g(\cdot)\rangle \ .$$

Similar to Lemmas II.1.4 and II.2.4, we obtain

$$(5.5) \qquad \lambda^{k+1}[S_{2\lambda}(f,k,t)-S_\lambda(f,k,t)] = \sum_{j=k+1}^{2k+2} Q(j,k,t)f^{(j)}(t)+o(1)$$

$$\equiv P_{2k+2}(D)f(t)+o(1)$$

where $Q(j,k,t)$ are polynomials depending only on the kernels $W(\lambda,t,u)$ of $S_\lambda(\cdot,t)$.

As with the real-valued cases, we can conclude from (5.5) that, if $f \in C^{2k+2}([a,b],X)$, $g \in C_o^\infty([0,\infty),X^*)$ with $\operatorname{supp} g \subset (a,b)$, the following relation holds:

$$(5.6)\qquad \lim_{\lambda\to\infty} <\lambda^{k+1}[S_{2\lambda}(f,k,\cdot)-S_{\lambda}(f,k,\cdot)],g(\cdot)>$$

$$= <P_{2k+2}(D)f(\cdot),g(\cdot)> = <f(\cdot),P^{*}_{2k+2}(D)g(\cdot)> \quad .$$

Following the same argument as before, we see it remains to justify the following interchange of limits:

$$(5.7)\qquad \lim_{\ell\to\infty}\ \lim_{\lambda_n\to\infty} <\lambda_n^{k+1}[S_{2\lambda_n}(f_\ell,k,\cdot)-S_{\lambda_n}(f_\ell,k,\cdot)],g(\cdot)>$$

$$= \lim_{\lambda_n\to\infty}\ \lim_{\ell\to\infty} <\lambda_n^{k+1}[S_{2\lambda_n}(f_\ell,k,\cdot)-S_{\lambda_n}(f_\ell,k,\cdot)],g(\cdot)>$$

where $f_\ell \in C^{2k}([a_1,b_1],X)$, $\operatorname{supp} g \subset (a_1,b_1) \subset [a_1,b_1] \subset (a,b)$ and $f_\ell \to f$.

The justification of (5.7) follows by a relation similar to (2.6), (2.14) and (2.25) (in Lemmas II.1.6 and II.2.5):

$$(5.8)\qquad |\lambda^{k+1}[S_{2\lambda}(f,k,\cdot)-S_{\lambda}(f,k,\cdot)],g(\cdot)>|$$

$$\leq M\,\|f\|_{L_N^{2k}([a_1,b_1],X)}$$

where

$$\|f\|_{L_N^{2k}([a_1,b_1],X)} \equiv \sup_{0\leq t<\infty} \|f(t)\|_X\, e^{-Nt} +$$

$$+ \max_{0\leq i\leq 2k} \|f^{(i)}\|_{C([a_1,b_1],X)} \quad .$$

The proof of (5.8) is also similar to (2.6), (2.14) and (2.25). We would like to remark that, although the proof for (5.8) would be quite involved if one actually carried it out in detail, the calculation depends mainly on the basic properties of the kernel $W(\lambda,t,u)$ of $S_\lambda(\cdot,t)$.

Therefore, we have the following slightly generalized version to facilitate the proof of Theorem V.3.1.

<u>Theorem V.4.1</u>

If $f \in C([0,\infty),X)$, $\|f(t)\|_X \leq M\, e^{Nt}$, $0 < a < a_1 < b_1 < b < \infty$, $\lambda_n \to \infty$ not faster than some geometric sequence and

$$I_\lambda(f,\lambda,k,a,b) = \lambda^{k+1}\|S^i(f,k,\cdot)-f(\cdot)\|_{C([a,b],X)} ,$$

then the following implication holds: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), and (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6), where

(1) $I_i(f,\lambda_A,k,a,b) = O(1)$;

(2) $f^{(2k+1)} \in A.C.((a,b),X)$ and $f^{(2k+2)} \in L_\infty([a,b],X)$;

(3) $I_i(f,\lambda,k,a,b) = O(1)$;

(4) $I_i(f,\lambda_n,k,a,b) = o(1)$;

(5) $f \in C^{2k+2}(a,b)$ and $\displaystyle\sum_{m=k+1}^{2k+2} Q_i(m,k,t)f^{(m)}(t) = 0$ in (a,b) where $Q_i(m,k,t)$ are polynomials depending on k and i ;

(6) $I_i(f,\lambda,k,a_1,b_1) = o(1)$.

As a consequence of Theorem V.4.1, we can conclude from

$$(5.9) \qquad \|H_i(\lambda_n,k,t)f-T(t)f\|_{C([a,b],X)} = O(\lambda_n^{-(k+1)})$$

that

$$(5.10) \qquad \frac{d^{2k+2}}{dt^{2k+2}} T(t)\, f \in L_\infty([a,b],X) \quad .$$

In particular, for any $\epsilon > 0$, there is a t^*, $a < t^* < a+\epsilon$, such that

$$\frac{d^{2k+2}}{dt^{2k+2}} T(t)f\Big|_{t=t^*}$$

exits. But it is well-known (see, e.g. [7], p. 9) that, if $f \in D(A^r T(s))$, then $\frac{d^r}{dt^r} T(t)f$ $(= A^r T(t)f)$ exists and is continuous for $t > s$. In other words, it follows from (5.10) that $T(t)f \in D(A^{2k+2})$ for $t > a$.

On the other hand, the same reasonings show that, if $T(t^*)f \in D(A^{2k+2})$, then

$$\|H_i(\lambda,k,t)f-T(t)f\|_{C([a,b],X)} = O(\lambda^{-(k+1)})$$

for any $t^* < a < b$.

In the case that

$$(5.11) \qquad \|H_i(\lambda_n,k,t)f-T(t)f\|_{C([a,b],X)} = o(\lambda_n^{-(k+1)}) \quad ,$$

by using Theorem V.4.1 again, we first conclude that

$$(5.12) \qquad \sum_{j=k+1}^{2k+2} Q(j,k,t)A^j T(t)f = 0, \quad t_o < t < t_o+\delta \ .$$

Since the A^j are closed operators, letting $t \to t_o$, we have $T(t_o)\, f \in D(A^{2k+2})$ and

$$\sum_{j=k+1}^{2k+2} Q(j,k,t_o)\, A^j T(t_o) f = 0 \quad .$$

§5 The Inverse Theorem

The inverse problem can be solved similarly, that is, by generalizing Theorem IV.1.4. We define an intermediate space $C_o(\alpha,k;a',b')$ consisting of functions f in $C_o([0,\infty),X)$ with support in $[a',b']$, and satisfying

$$\|f\|_\alpha \equiv \sup_{0<\xi<1} \xi^{-\alpha/2} K(\xi,f) < \infty \quad ,$$

where

$$(5.13) \qquad K(\xi,f) = \inf_{g \in G} \{\|f-g\|_{C([a,b],X)} + \xi[\|f\|_{C([a,b],X)} + \|g^{(2k+2)}\|_{C([a,b],X)}]\} \,, \qquad 0 < \xi \le 1$$

and

$$(5.14) \qquad G = \{g; g \in C_o^{2k+2}([0,\infty),X),\ \operatorname{supp} g \subset [a',b']\} \quad .$$

Using arguments similar to those of Lemmas IV.2.1, IV.2.2 and IV.2.3, we have, for functions f in $C_o([0,\infty),X)$ with supports in (a',b'), that

$$(5.15) \qquad \|S_\lambda(f,k,t)-f(t)\|_{C([a,b],X)} \le M^{\frac{\alpha}{2}(k+1)} \,, \quad 0 < \alpha < 2$$

is equivalent to $f \in C_o(\alpha,k;a',b')$.

Following Theorem IV.3.1, we obtain the equivalence of $f \in C_o(\alpha,;a',b')$ and $f \in \mathrm{Liz}\,(\alpha,k+1;a,b)$ for such functions (i.e., supp $f \in (a',b')$). This completes the proof of the special case.

The general case can be deduced from the special case by a similar argument used in Section 4 of Chapter IV. In this case, the corresponding function g (by considering the produce fg we reduce the orginal case to the case of functions with compact supports) is a "scalar-valued" function, that is, $g \in C_o^\infty([0,\infty),k)$. The rest of the proof is carried out analogeously.

Therefore, we have the following generalized version of Theorem IV.1.4:

<u>Theorem V.5.1</u>

Let $0 < a_i < a_{i+1} < b_{i+1} < b_i < \infty$, $i = 1,2$, $0 < \alpha < 2$, $f \in C([0,\infty),X)$, X a relexive Banach space, with $\|f(t)\|_X \leq M\,e^{Nt}$. Then in the following, the implication (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) holds:

$$(1)\quad \|S_\lambda^i(f,k,t)-f(t)\|_{C([a_1,b_1],X)} = 0(\lambda^{-\frac{\alpha}{2}(k+1)})$$

$$(2)\quad f \in \mathrm{Liz}\,(\alpha,k+1;C([a_2,b_2],X))\quad ,$$

$$(3)\quad \|S_\lambda^i(f,k,t)-f(t)\|_{C([a_3,b_3],X)} = 0(\lambda^{-\frac{\alpha}{2}(k+1)})\quad .$$

By applying the above theorem, we can deduce from

$$(5.16)\qquad \|H_i(\lambda_n,k,t)f-T(t)f\|_{C([a,b],X)} = 0(\lambda_n^{-\frac{\alpha}{2}(k+1)})$$

that

(5.17) $$T(t)f \in \text{Liz}\,(\alpha,k+1;C([a',b'],X))$$

for all $[a',b'] \subset (a,b)$. That is, for t, $t+k\tau \in [a'b']$, $|\tau| < h$, there holds

(5.18) $$\|\Delta_\tau^{2k+2}\ T(t)f\|_X \leq M\ h^{\alpha(k+1)} \quad .$$

To show $f \in \text{Liz}_T\,(\alpha,k+1;t_o)$, it is sufficient to show for every $\xi > t_o$, $\omega_T^{2k+2}(f,h;\xi) = 0(h^{\alpha k})$.

Let $\xi > t_o$ be given. Let $t_o < a < b$ such that

(5.19) $$\lambda^{\frac{\alpha}{2}(k+1)}\ \|H_1(\lambda,k,t)-T(t)f\|_{C([a,b],X)} \leq M\ (a,b) \quad .$$

Then choose $[a',b'] \subset (a,b)$ such that $a' < \xi$. Finally choose $h > o$ be so cmall such that $(2k+2)h \leq \min\,(\xi-a',b'a')$.

Now (5.19) implies

$$\|\Delta_\tau^{2k+2}\ T(t)f\|_X \leq M\ h^{\alpha(k+1)} \quad ,$$

$t,t+k\tau \in [a',b']$. Let M' be a constant such that

$$M' \geq \sup_{0\leq t\leq \xi+1} \|T(t)\| \ ,$$

such M' does exist by Proposition 1.1.2 of ([7], p.8). For $\xi \leq t \leq \xi+1$, we estimate $\|\Delta_\tau^{2k+2}\ T(t)\|_X$ as follows:

(i) $0 < \tau \leq h$

$$\|\Delta_\tau^{2k+2}\ T(t)f\|_X \leq \|T(t-a')\| \cdot \|\Delta_\tau^{2k+2}\ T(a')f\|_X$$

$$\leq M' \cdot M\ h^{\alpha(k+1)} \quad .$$

(ii) $-h < \tau < 0$

$$\|\Delta_{\tau}^{2k+2}\ T(t)f\|_X$$

$$\leq \|T(t+(2k+2)\delta-a')\| \cdot \|\Delta_{-\delta}^{2k+2}(a')f\|$$

$$\leq M' \cdot M\, h^{\alpha(k+1)}$$

Therefore,

$$\text{(5.20)} \qquad \omega_T^{2k+2}\ (f,h;\xi) = O(h^{\alpha(k+1)})\ , \qquad h \to 0^+ \ .$$

In other words, $f \in \mathrm{Liz}_T\ (\alpha,k+1;t_o)$.

The converse deducing

$$\|H_1(\lambda,k,t)f-T(t)f\|_{C([a,b],X)} = O(\lambda^{-\frac{\alpha}{2}(k+1)})$$

from $f \in \mathrm{Liz}_T\ (\alpha,k+1;t_o)$ for any $t_o < a < b$ is trivial. The details of the proof in this part will be omitted.

§6. An Example

In the theorems just proved, we use the estimate of $\|H(\lambda,k,t)f-T(t)f\|$ for t in an interval. It is tempting to try proving these theorems by using the estimate of $\|H(\lambda,k,t)f_o-T(t_o)\|$ only.

However, we have given a counter example in [22] for which

$$\|H_2(\lambda_n,t_o)f-T(t_o)f\|_X = O(\lambda_n) \quad ,$$

where

$$H_2(\lambda_n,t) \equiv H_2(2^{2^{n+1}},t) \equiv H_2(2^{2^{n+1}},0,t)$$

is Post-Widder's operator (Definition V.2.1, equation (5.1)), but

$$\overline{\lim_{h \to o}} \left\|\frac{1}{h^2} \{T(t_o-h)f-2T(t_o)f+T(t_o+h)f\}\right\|_X = \infty \ .$$

In fact let $X = C_o[0,\infty)$, $T(t)f(x) = f(x+t)$ and let

$$f(x) = \sum_{m=3}^{\infty} f_{2^m}(x) \quad ,$$

where

$$(5.20) \qquad f_m(x) = \begin{cases} 2^{-m}\left(1 - \dfrac{x-2^{-1}+2^{-m}}{2^{-3m}}\right) , & x-2^{-1}+2^{-m} K 2^{-3m} \\ \\ 0 & \text{otherwise} \ . \end{cases}$$

Obviously, when $h = 2^{-2^n}$ and $t_o = \frac{1}{2}$, $T(\frac{1}{2})f = T(\frac{1}{2} + h)f = 0$ and

$$\|h^{-2}\{T(t_o-h)f-2T(t_o)f+T(t_o+h)f\|_X \geq 2^{2^{n+1}} \cdot 2^{-2^n} = 2^{2^n}$$

which is not bounded. On the other hand, we claim

$$\|H_2(2^{2^{n+1}},\tfrac{1}{2})f-T(\tfrac{1}{2})f\|_X \quad \text{is} \quad 0(2^{-2^{n+1}}) \quad .$$

First, we write $H_2(2^{2^{n+1}},\frac{1}{2})f$ be a sum of three terms:

$$(5.21) \qquad H_2(2^{2^{n+1}},\tfrac{1}{2})f = \sum_{m=3}^{\infty} H_2(2^{2^{n+1}},\tfrac{1}{2})f_{2^m} \equiv \sum_{m=3}^{n+1} + \sum_{m=n}^{\infty} + \sum_{m=n+1}^{\infty}$$

$$\equiv I_1 + I_2 + I_3 \ .$$

Then we estimate $\|I_1\|_X$, $\|I_2\|_X$ and $\|I_3\|_X$ separately:

$$\|I_1\|_X \le \sup_x \left| \sum_{k=0}^{k_1} \binom{2^{2^{n+1}}}{k} \left(\tfrac{1}{2}\right)^{2^{2^{n+1}}} T\left(\frac{k}{2^{2^{n+1}}}\right) \sum_{m=3}^{n+1} f_{2^m}(x) \right|$$

where $k_1 = \max\{k;\ k \le (2^{-1} - 2^{-2^{n-1}} + 2^{-3\cdot 2^{n-1}}) 2^{2^{n-1}}\}$, or, since

$$\left\| \sum_{m=3}^{n-1} f_{2^m}(x) \right\| \le 1 \ ,$$

following ([30], p. 15 (8)) for $\alpha = \frac{1}{4}$, we have $\|I_1\|_X \le M_1 \ 2^{-2^{n+1}}$.

The estimate of $\|I_3\|_X$ is easy:

$$\|I_3\|_X \le \sup_x \left| H_2\left(2^{-2^{n+1}}, \tfrac{1}{2}\right) \sum_{m=n+1}^{\infty} f_{2^m} \right| \le M \sup_x \sum_{m=n+1}^{\infty} |f_{2^m}|$$

$$\le 2M \cdot 2^{-2^{n+1}} \ , \qquad (M = \sup_{0 \le t \le 1} \|T(t)\|) \ .$$

Finally, we estimate

$$\|I_2\|_X \le \sup_x \left| \sum_{k=0}^{k_2} \binom{2^{2^{n+1}}}{k} \left(\tfrac{1}{2}\right)^{2^{2^{n+1}}} f_{2^n}\left(\frac{k}{2^{2^{n+1}}} + x\right) \right|$$

where $k_2 = \max\{k;\ k \le (2^{-1} - 2^{-2^n} + 2^{-3\cdot 2^n})\ 2^{2^{n+1}}\}$. Since the

$$\mathrm{supp}\ f_{2^n}\left(\frac{k}{2^{2^{n+1}}} + x\right)$$

are distinct, we conclude

$$||I_2|| \le \binom{2^{2^{n+1}}}{k_2} \cdot \left(\frac{1}{2}\right)^{2^{2^{n+1}}} \cdot 2^{-2^n} \le M_2 \, 2^{-2^{n+1}} .$$

Therefore

$$||H_2(2^{2^{n+1}}, \tfrac{1}{2})f - T(\tfrac{1}{2})f||_X = ||H_2(2^{2^{n+1}}, \tfrac{1}{2})f||_X$$

$$\le M' \, 2^{-2^{n+1}} .$$

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